Selberg integrals and the AGT conjecture

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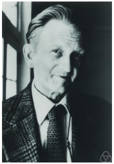
For $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, Euler (1738) proved the beta integral evaluation

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

where for $z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$,

$$\begin{aligned} \Gamma(z) &:= \lim_{n\to\infty} \frac{n!\, n^{z-1}}{z(z+1)\dots(z+n-1)} && z \notin \{0,-1,-2,\dots\} \\ &= \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t && \mathrm{Re}(z) > 0, \end{aligned}$$

is the gamma function.



In 1941/1944 Atle Selberg discovered a multidimensional analogue of Euler's beta integral

$$\int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i - t_j|^{2\gamma}\, \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

$$= \prod_{i=1}^{k} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(k+i-2)\gamma)\Gamma(1+\gamma)},$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ and

$$\mathrm{Re}(\gamma) > -\min\left\{\frac{1}{k}, \frac{\mathrm{Re}(\alpha)}{k-1}, \frac{\mathrm{Re}(\beta)}{k-1}\right\}.$$

Three generalisations

Henceforth if $t = (t_1, \ldots, t_k)$ then $\mathrm{d}t := \mathrm{d}t_1 \cdots \mathrm{d}t_k$ and

$$\Delta(t) = \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j).$$

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1. The Kadell integral: Let $P_\lambda^{(1/\gamma)}(t)$ be a Jack polynomial, then

$$\int_{[0,1]^k} P_\lambda^{(1/\gamma)}(t) |\Delta(t)|^{2\gamma} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \, \mathrm{d}t$$

$$= P_\lambda^{(1/\gamma)}(\underbrace{1, \ldots, 1}_{k \text{ times}}) \prod_{i=1}^{k} \frac{\Gamma(\alpha + (k-i)\gamma + \lambda_i)\Gamma(\beta + (i-1)\gamma)\Gamma(1 + i\gamma)}{\Gamma(\alpha + \beta + (2k-i-1)\gamma + \lambda_i)\Gamma(1+\gamma)},$$

where $\mathrm{Re}(\alpha) > -\lambda_k$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > \ldots$.

This was conjectured by Macdonald and subsequently proved by Kadell in 1987.

2. The Hua–Kadell integral:

$$\int_{[0,1]^k} P_\lambda^{(1/\gamma)}(t) P_\mu^{(1/\gamma)}(t) \, |\Delta(t)|^{2\gamma} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\gamma-1} \, \mathrm{d}t$$

$$= P_\lambda^{(1/\gamma)}(\underbrace{1,\dots,1}_{k \text{ times}}) P_\mu^{(1/\gamma)}(\underbrace{1,\dots,1}_{k \text{ times}})$$

$$\times \prod_{i=1}^{k} \frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\gamma+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\gamma+(k-i-1)\gamma+\lambda_i)\Gamma(1+\gamma)}$$

$$\times \prod_{i,j=1}^{k} \frac{\Gamma(\alpha+\gamma+(2k-i-j-1)\gamma+\lambda_i+\mu_j)}{\Gamma(\alpha+\gamma+(2k-i-j)\gamma+\lambda_i+\mu_j)},$$

which is a generalisation of Kadell's integral, but with the added restriction $\beta = \gamma$. In the above $\mathrm{Re}(\alpha) > -\lambda_k$, $\mathrm{Re}(\gamma) > 0$.

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Proved by Hua for $\gamma = 1$ (Schur case) in 1979, and in general by Kadell in 1993.

3. The Alba–Fateev–Litvinov–Tarnopolskiy (AFLT) integral:

$$\int\limits_{[0,1]^k} P_\lambda^{(1/\gamma)}(t) P_\mu^{(1/\gamma)}[t+\beta/\gamma-1]\,|\Delta(t)|^{2\gamma} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\,\mathrm{d}t$$

$$= P_\lambda^{(1/\gamma)}[k] P_\mu^{(1/\gamma)}[k+\beta/\gamma-1]$$

$$\times \prod_{i=1}^{k} \frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-\ell-i-1)\gamma+\lambda_i)\Gamma(1+\gamma)}$$

$$\times \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{\Gamma(\alpha+\beta+(2k-i-j-1)\gamma+\lambda_i+\mu_j)}{\Gamma(\alpha+\beta+(2k-i-j)\gamma+\lambda_i+\mu_j)},$$

where ℓ is an arbitrary integer such that $\ell \geqslant l(\mu)$. This is the Hua–Kadell integral which removes the restriction $\beta = \gamma$.

Discovered by AFLT in 2010, and proved for $\alpha = N\gamma$, $\beta = M\gamma$ where $N, M \in \mathbb{N}$ and $\mathrm{Re}(\gamma) \geqslant 0$.

The AGT Conjecture

The motivation of Alba, Fateev, Litvinov and Tarnopolskiy was the verification of the so-called AGT conjecture for SU(2).

An ingredient in the AGT conjecture is an explicit expression for the Nekrasov partition function in terms of conformal blocks in Liouville field theory.

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In order to verify this expression, AFLT looked for an orthogonal basis for the space of representations of $\mathrm{Vir} \oplus \mathscr{A}$, where Vir is the Virasoro algebra and $\mathscr{A}$ is the Heisenberg algebra.

This boils down to computing the AFLT integral!

Higher rank integrals

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Here, the ordinary Selberg integral is associated to the Lie algebra A_1, with a single set of k integration variables:

This gives the factor

$$|\Delta(t)|^{2\gamma} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}$$

For A_2 the picture is:

Given a pair of integers $k \leqslant \ell$ the corresponding integrand is

$$\mathcal{I}^{k,\ell}(t,u;\gamma) := |\Delta(t)|^{2\gamma}|\Delta(u)|^{2\gamma}|\Delta(t,u)|^{-\gamma}\prod_{i=1}^{k} t_i^{\alpha_1-1}\prod_{i=1}^{\ell} u_i^{\alpha_2-1}(1-u_i)^{\beta-1},$$

where the adjacent vertices are paired by the factor

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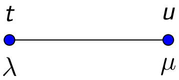
The exponents of the Vandermonde-type products come from the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The A_2 AFLT integral is therefore

$$\int_{C^{k,\ell}[0,1]} P_\lambda^{(1/\gamma)}(t) P_\mu^{(1/\gamma)}[u+\beta/\gamma-1]\,\mathcal{I}^{k,\ell}(t,u;\gamma)\,\mathrm{d}t\,\mathrm{d}u.$$

Or thinking about this again in terms of the Dynkin diagram:



In general, we can evaluate the A_n AFLT integral of the above form. The proof uses Macdonald polynomial theory, in particular generalisations of

$$\sum_\lambda P_\lambda(X;q,t) Q_\lambda(Y;q,t) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$.

Further A_n integrals

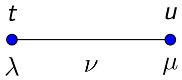
Also in 2010, Matsuo and Zhang were inspired by the AGT conjecture to consider AFLT-type Selberg integrals of the form

$$\int_{\mathcal{C}} s_\lambda(t) s_\nu[u-t] s_\mu[u+\beta-1]\, \mathcal{I}^{k,\ell}(t,u;1)\,\mathrm{d}t\,\mathrm{d}u.$$

Note that this is an analogue of the AFLT integral but with $\gamma = 1$, so that the Jack polynomials reduce to Schur functions:

$$P_\lambda^{(1)}(t) = s_\lambda(t) = \frac{\det_{1\leqslant i,j\leqslant k}\big(t_i^{\lambda_j+k-j}\big)}{\Delta(t)}.$$

We can dress up the Dynkin diagram futher:



The function $s_\nu[u - t]$ can be explained using plethystic notation. The power sum symmetric functions

$$p_r(X) = x_1^r + x_2^r + x_3^r + \cdots$$

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$$p_r[X + Y] = p_r[X] + p_r[Y].$$

The difference is then defined by

$$p_r[X - Y] := p_r[X] - p_r[Y],$$

so that

$$p_r[X + Y - Y] = p_r[X] + p_r[Y] - p_r[Y] = p_r[X]$$

as it should! As these are defined in terms of an algebraic basis, we can extend to any symmetric function.

We are able to evaluate the integral

$$\left\langle s_\lambda(t)s_\nu[u-t]s_\mu[u+\beta-1]\right\rangle = \int_{\mathcal{C}} s_\lambda(t)s_\nu[u-t]s_\mu[u+\beta-1]\,\mathcal{I}^{k,\ell}(t,u;1)\,\mathrm{d}t\,\mathrm{d}u,$$

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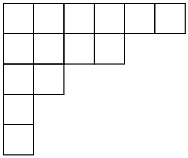
2. The second method is based on a new integral formula for complex Schur functions, which allows for a proof for general n by induction on the rank.

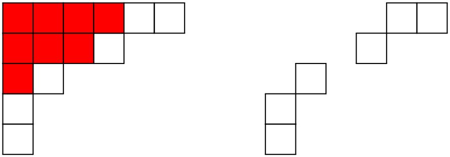
Caveat: method (1) works for $n = 2$ only.

Partitions

We identify a partition with its Young diagram:

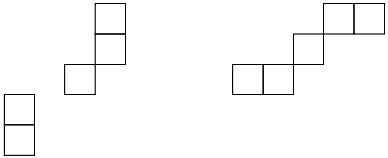


which represents $(6, 4, 2, 1, 1)$. We write $\mu \subseteq \lambda$ if the diagram of μ is contained in that of λ. For example



so $(4, 3, 1) \subseteq (6, 4, 2, 1, 1)$. On the right is the resulting skew shape $(6, 4, 2, 1, 1)/(4, 3, 1)$.

We say λ/μ is a vertical strip if it contains at most one box in each row, and a horizontal strip if it has at most one box in each column:



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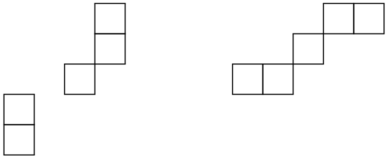

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Finally, we define the complete symmetric functions by

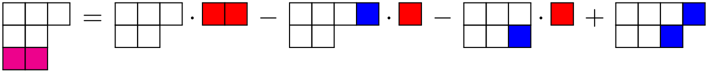
$$h_r(X) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r} x_{i_1} \cdots x_{i_r} = s_{(r)}(X),$$

the sum over all monomials of degree r on the alphabet X.

The inverse Pieri rule may be stated as

$$s_{(\nu,d)}[u-t] = \sum_{\substack{\omega' \succ \nu' \\ l(\omega)=l(\nu)}} (-1)^{|\omega/\nu|} s_{\omega}[u-t] h_{d-|\omega/\nu|}[u-t].$$

At the level of diagrams:



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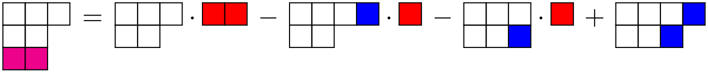

This, together with some plethystic magic, leads to a recursion of the form

$$\left\langle s_\lambda(t) s_{(\nu,d)}[u-t] s_\mu[u+\beta-1] \right\rangle$$
$$= \sum_{\substack{\eta' \succ \lambda' \\ \pi' \succ \mu'}} \sum_{\substack{\omega' \succ \nu' \\ l(\nu)=l(\omega)}} \left\langle s_\eta(t) s_\omega[u-t] s_\pi[u+\beta-1] \right\rangle.$$

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