GARSIA–HAIMAN MODULES

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In his 1988 paper [6] Ian Macdonald introduced a remarkable new class of symmetric functions that now bear his name. These polynomials encompass many other classes of symmetric functions as special cases; namely the Schur polynomials, Hall–Littlewood polynomials, and Jack symmetric functions. Each of these classes have important applications in representation theory, and it turns out that this also holds for the Macdonald polynomials. In this report we will give an outline of the representation-theoretic interpretation of the Macdonald polynomials.

Let Λ_n be the *ring of symmetric functions* in the countable alphabet $X = (x_1, \dots, x_n)$ over the field $\mathbb{Q}(q,t)$ of rational functions in independent indeterminates q and t. The *monomial symmetric functions* are defined by $m_\lambda(X) = \sum_\alpha x^\alpha$ where the sum is over all distinct permutations of the partition λ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Similarly the *power sum symmetric functions* are given by $p_r(X) = x_1^r + \cdots + x_n^r$ for $r \in \mathbb{N}$ and generalised to partitions by $p_\lambda(X) = p_{\lambda_1}(X) \cdots p_{\lambda_n}(X)$. When indexed over all partitions λ with $l(\lambda) \leqslant n$ both $\{m_\lambda\}$ and $\{p_\lambda\}$ form bases for Λ_n.

Symmetric functions arise naturally in the representation theory of the symmetric group $\mathfrak{S}_n$. Denote by z_μ the size of the $\mathfrak{S}_n$ conjugacy class indexed by μ. Then writing $\mu \vdash n$ for μ a partition of n we have for χ a character of $\mathfrak{S}_n$ that the *Frobenius characteristic* is given by

$$\Phi(\chi) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_\mu p_\mu(X), \tag{1}$$

(see [7, p. 167]). Note that χ_μ is the value of χ on the conjugacy class indexed by μ. Each partition of n corresponds to an irreducible representation of $\mathfrak{S}_n$, denoted χ^λ. Under the Frobenius characteristic these map to the most important class of symmetric functions, the *Schur functions*, written $\Phi(\chi^\lambda) = s_\lambda(X)$. The Schur functions have a nice expansion in the monomial basis,

$$s_\lambda(X) = \sum_{\mu \leqslant \lambda} K_{\lambda\mu} m_\mu(X).$$

Here $\mu \leqslant \lambda$ means μ is less than λ in the dominance partial order on partitions. The coefficients $K_{\lambda\mu}$ are called the Kostka numbers, and play an important role in representation theory, combinatorics, and the theory of symmetric functions. In [6] Macdonald found a similar expansion for the Macdonald polynomials in terms of modified Schur functions which for brevity we will not state here. He denoted these coefficients $K_{\lambda\mu}(q,t)$ and observed that when $(q,t) = (0,1)$ they reduced to the Kostka numbers. After computing these coefficients by hand for $n \leqslant 7$ he then made the following conjecture.

Theorem 1 (Macdonald Positivity Conjecture)**.** *The Kostka–Macdonald coefficients are polynomials in q and t with nonnegative integer coefficients.*

The proof that the $K_{\lambda\mu}(q,t)$ are indeed polynomial with integral coefficients was established independently by several authors during the 1990s. The positivity however took much longer to prove, finally being shown by Mark Haiman in 2001 [5]. In order to do so Haiman (joint with Adriano Garsia in earlier work) worked with the *transformed Macdonald polynomials.* These have a much more pleasant expansion in terms of ordinary Schur functions,

$$\tilde{H}_\mu(X; q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t) s_\lambda(X),$$

where $\tilde{K}_{\lambda\mu}(q,t)$ is a simple multiple of $K_{\lambda\mu}(q,t)$ for which the positivity conjecture still holds.

Setting $q = 0$ in the Kostka–Macdonald coefficients yields the Kostka–Foulkes polynomial $K_q(t)$, which also have nonnegative integer coefficients. In 1992 Adriano Garsia and Claudio Proseci proved this result using the representation theory of the symmetric group, hoping that this approach would generalise to Macdonald's case [2]. In order to correctly adapt this approach Garsia and Haiman introduced particular doubly graded $\mathfrak{S}_n$ modules which we now describe. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two families of independent indeterminates and let $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the ring of polynomials in $2n$ variables. For any $w \in \mathfrak{S}_n$ the diagonal action of the symmetric group on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ is given by

$$wf(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)}).$$

The space $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ is doubly graded by degree, so $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \bigoplus_{r,s} \mathbb{C}[\mathbf{x}, \mathbf{y}]_{r,s}$ with r, s the x-degree and y-degree respectively. Furthermore the diagonal action of $\mathfrak{S}_n$ respects the grading. Let μ be a partition, and consider its Young diagram, formed by placing μ_i squares in the ith row. This can be described as $\mu = \{(i, j) : j < \mu_{i+1}\} \subset \mathbb{N}^2$ where we index from zero. One forms a two-variable analogue of the famous Vandermonde determinant by

$$\Delta_\mu = \det(x_k^j y_k^i)_{\substack{1 \leqslant k \leqslant n \\ (i,j) \in \mu}}.$$

Using this Garsia and Haiman construct the space

$$D_\mu = \mathbb{C}[\partial \mathbf{x}, \partial \mathbf{y}]\Delta_\mu.$$

This is the subspace of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ spanned by all partial derivatives of Δ_μ. We call D_μ the *Garsia–Haiman module* indexed by the partition μ. As the polynomial Δ_μ is $\mathfrak{S}_n$-alternating, it follows that D_μ is an $\mathfrak{S}_n$-invariant subspace and hence an $\mathfrak{S}_n$-module. Garsia and Haiman then proposed in [3] the following seemingly unremarkable result.

Theorem 2 ($n!$ Conjecture)**.** *For μ a partition of n the dimension of D_μ is $n!$.*

This result became the key that unlocked the proof of the positivity conjecture. The dimension of D_μ being $n!$ would imply that D_μ admits the regular representation, and so in light of the results below we can view D_μ as a q, t-refinement of the regular representation of $\mathfrak{S}_n$. The theorem resisted any attempt at an elementary proof, and it took almost ten years until Haiman proved the result using techniques from algebraic geometry.

In order to relate the transformed Macdonald polynomials to D_μ we need a doubly graded version of (1)Similarly for the doubly graded $\mathfrak{S}_n$ module D_μ one defines the *Frobenius series* to be

$$\mathcal{F}_{D_\mu}(X; q, t) = \sum_{r,s} \Phi(\mathrm{ch}(D_\mu)_{r,s}) t^r q^s$$

where $\mathrm{ch}(D_\mu)_{r,s}$ is the character of the submodule $(D_\mu)_{r,s}$. Haiman conjectured the following interpretation of the transformed Macdonald polynomials.

Theorem 3. *The bigraded Hilbert series of D_μ coincides with the transformed Kostka–Macdonald coefficients, written as*

$$\tilde{K}_{\lambda\mu}(q, t) = \sum_{r,s} \mathrm{mult}(\chi^\lambda, \mathrm{ch}(D_\mu)_{r,s}) t^r q^s.$$

This is equivalent to

$$\mathcal{F}_{D_\mu}(X; q, t) = \tilde{H}_\mu(X; q, t).$$

It is easy to see that if this interpretation holds, then the positivity conjectured by Macdonald also holds. Haiman showed in [4] that Theorem 2 together with some results from algebraic geometry implies Theorem 3, which in turn implies Theorem 1. Haiman's final proof was a significant achievement in theory of symmetric functions, however his method was not strictly combinatorial. A purely combinatorial proof of the conjecture came later in his combined work Ian Grojnowski through the theory LLT (Lascoux, Leclerc and Thibon) polynomials in an unpublished manuscript.

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