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Generalised Selberg Integrals and Macdonald Polynomials

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Die liebe Erde allüberall blüht auf im Lenz
und grünt aufs neu! Allüberall
und ewig blauen licht die Fernen!
Ewig... ewig...

The dear earth everywhere blossoms in spring
and grows green anew! Everywhere
and forever, blue is the horizon!
Forever... forever...

—Gustav Mahler, *Der Abschied*
from *Das Lied von der Erde*

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 The Selberg Integral

In 1944 Atle Selberg proved the following remarkable multiple integral [Sel44]. For k a positive integer

$$\begin{aligned} S_k(\alpha,\beta,\gamma) &:= \int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i-t_j|^{2\gamma}\,\mathrm{d}t \\ &= \prod_{j=1}^{k} \frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(k+j-2)\gamma)\Gamma(1+\gamma)}, \end{aligned} \tag{1.1.1}$$

where $\mathrm{d}t$ stands for $\mathrm{d}t_1\cdots\mathrm{d}t_k$. This is valid for complex parameters α,β with strictly positive real parts and complex γ satisfying

$$\mathrm{Re}(\gamma) > -\min\{1/k, \mathrm{Re}(\alpha)/(k-1), \mathrm{Re}(\beta)/(k-1)\}.$$

The integral and its evaluation has come to be known as the Selberg integral, and has appeared in numerous areas of mathematics, such as random matrix theory [For10, Chapter 3], analytic number theory [KS00], and conformal field theory [AFLT11, MZ11]. The Selberg integral is a generalisation of Euler's beta integral, and reduces to it when $k=1$:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.1.2}$$

Here α and β are again complex numbers with strictly positive real parts. The original proof of (1.1.1) by Selberg relied on the assumption that γ is a positive integer so that the Vandermonde-type product may be expanded as a polynomial. Iterating the beta integral then gives the result. For the case of general γ, one may use Carlson's theorem (see [And86, p. 51]) to analytically continue γ off the integers and achieve the desired evaluation. It appears Selberg did not know of Carlson's theorem, but did prove an equivalent result to arrive at the same conclusion. Whilst Selberg's proof was reasonably elegant, many simpler methods of proof have since been discovered. For a comprehensive account refer to [AAR99, Chapter 8] or [FW08].

The Selberg integral remained largely unknown to the wider mathematical community for a period of over three decades. Integrals of Selberg type then began to arise naturally in the study of constant term identities by Dyson

[Dys62]. One such integral is the following exponential form of the Selberg integral:

$$\frac{1}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} \exp\left(-\frac{1}{2}\sum_{i=1}^{k} t_i^2\right) \prod_{1\leqslant i<j\leqslant k} |t_i - t_j|^{2\gamma}\,\mathrm{d}t = \prod_{j=1}^{k} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}, \qquad (1.1.3)$$

the evaluation of which is valid for $\mathrm{Re}(\gamma) > -1/k$. One may obtain (1.1.3) from the Selberg integral (1.1.1) by setting $\alpha = \beta$, making the substitution $t_i \mapsto (1 + t_i/\sqrt{2\alpha})/2$, and taking the limit $\alpha \to \infty$ with the aid of Stirling's formula [AAR99, Corollary 8.2.3]. The integral was conjectured by Dyson and Mehta [MD63]. However, neither of them knew of the Selberg integral, and so this method of proof was not accessible. Eventually through investigations by Bombieri, Selberg was made aware of Mehta and Dyson's conjecture. He immediately observed that it could be obtained from (1.1.1) as described above. In 1982 Macdonald noticed that Mehta's integral may be associated with the reflection group A_{n-1} by virtue of the Vandermonde determinant occurring in the integrand [Mac82]. In the same article, he conjectured that an exponential integral of Selberg type exists for each finite reflection group. For the infinite families of types B_n and D_n Macdonald's conjectures may be obtained from the Selberg integral in a similar manner as type A_n. Opdam provided a uniform proof of these conjectures in 1989 using hypergeometric shift operators [Opd89].

In 1986 Macdonald made another conjecture related to the Selberg integral, this time involving the multiplication of the integrand by a symmetric function. Let $P_\mu^{(1/\gamma)}(t)$ be the Jack polynomial indexed by the partition μ in the variables $t = (t_1, \ldots, t_k)$ with parameter $1/\gamma$ (Definition 3.5 below). Macdonald conjectured the integral evaluation [Mac87, Conjecture C5]

$$\int_{[0,1]^k} P_\mu^{(1/\gamma)}(t) \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i - t_j|^{2\gamma}\,\mathrm{d}t \qquad (1.1.4)$$

$$= P_\mu^{(1/\gamma)}(\underbrace{1,1,\ldots,1}_{k\text{ times}}) \prod_{i=1}^{k} \frac{\Gamma(\beta+(i-1)\gamma)\Gamma(\alpha+(k-i)\gamma+\mu_i)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-i-1)\gamma+\mu_i)\Gamma(1+\gamma)}$$

This is known as Kadell's integral as they were the first to provide a proof [Kad97]. When the partition λ is a single column the Jack polynomial reduces to the rth elementary symmetric function, which is a variant of the Selberg integral due to Aomoto [Aom87]. Aomoto used this integral to provide an elementary proof of the Selberg integral. It turns out that the Kadell integral may in fact be generalised further with the inclusion of two Jack polynomials in the integrand. With the use of plethystic notation as defined in Section 2.3

the evaluation may be stated as

$$\int_{[0,1]^k} P_\mu^{(1/\gamma)}(t) P_\nu^{(1/\gamma)}[t+\beta/\gamma-1] \tag{1.1.5}$$
$$\times \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i-t_j|^{2\gamma}\,\mathrm{d}t$$
$$= P_\mu^{(1/\gamma)}[k] P_\nu^{(1/\gamma)}[k+\beta/\gamma-1]$$
$$\times \prod_{i=1}^{k} \frac{\Gamma(\beta+(i-1)\gamma)\Gamma(\alpha+(k-i)\gamma+\mu_i)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-\ell-i-1)\gamma+\mu_i)\Gamma(1+\gamma)}$$
$$\times \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{\Gamma(\alpha+\beta+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma(\alpha+\beta+(2k-i-j)\gamma+\mu_i+\nu_j)},$$

where μ and ν are partitions with $\ell(\mu) \leqslant k$ and $\ell(\nu) \leqslant \ell$ respectively. This integral was discovered by Alba, Fateev, Litvinov and Tarnopolskiy in their studies of conformal field theory. In this thesis we obtain a proof of the Alba–Fateev–Litvinov–Tarnopolskiy (AFLT) integral using the theory of Macdonald polynomials. This circumvents an issue in the original proof of Alba *et al.* which assumes some unnecessary restrictions on the parameters occurring in the integral. It should be noted that when $\beta=\gamma$ the above integral (1.1.5) was discovered independently by Hua in the case $\gamma=1$ and for general γ by Kadell [Hua79, Kad93] Hence when $\beta=\gamma$ (1.1.5) is called the Hua–Kadell integral.

1.2 q-Selberg Integrals and Macdonald Polynomials

A q-analogue (sometimes q-deformation) of a mathematical object is a generalisation of said object involving a new parameter q. In the limit $q\to 1$ the original object is returned. One important q-analogue is the q-integral or Jackson integral

$$\int_0^1 f(t)\,\mathrm{d}_q t := (1-q)\sum_{k=0}^{\infty} f(q^k)q^k$$

where f is a function for which the right hand side converges. Here q is assumed to be a real number such that $0<q<1$. This may be generalised to multidimensional q-integrals, and this is discussed in Section 4.2. Let us write $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$ for the infinite q-shifted factorial. For a complex number z we also define

$$(a;q)_z = \frac{(a;q)_\infty}{(aq^z;q)_\infty}.$$

In 1980 Aksey conjectured three different q-analogues of Selberg's integral [Ask80b]. The most foundational of these is now known as the Askey–Habsieger–

Kadell integral, and is given by

$$\begin{aligned}\int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(qt_i;q)_{\beta-1} \prod_{1\leqslant i<j\leqslant k} t_j^{2\gamma}(q^{1-\gamma}t_i/t_j;q)_{2\gamma}\,\mathrm{d}_q t \\ = q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} \prod_{i=1}^{k} \frac{\Gamma_q(\alpha+\gamma(i-1))\Gamma_q(\beta+\gamma(i-1))\Gamma_q(1+\gamma i)}{\Gamma_q(\alpha+\beta+\gamma(k+i-2))\Gamma_q(1+\gamma)},\end{aligned} \tag{1.2.1}$$

where $\Gamma_q(x) = (q;q)_{x-1}(1-q)^{1-x}$ is the q-gamma function. The integral was proved independently by Habsieger [Hab88] and Kadell [Kad88a]. Kadell continued to investigate these q-Selberg integrals in search of a generalisation of (1.1.4). Such a q-analogue involved a new two-parameter deformation of the Jack polynomials. The Jack polynomials themselves are a one-parameter deformation of the famous Schur functions, important in algebraic combinatorics and representation theory. Kadell dubbed his generalised Jack polynomials the Selberg polynomials and conjectured a corresponding Kadell-type integral evaluation at the q-level [Kad88b, Conjecture 4].

In the same year Macdonald introduced a new two-parameter family of symmetric functions generalising the Jack and Schur functions. Macdonald's functions coincided with Kadell's Selberg polynomials and are denoted by $P_\lambda(x;q,t)$. The Macdonald polynomials are elements of the ring of symmetric functions over the field $\mathbb{F} = \mathbb{Q}(q,t)$, written $\Lambda_{\mathbb{F}}$. In fact they form a basis for $\Lambda_{\mathbb{F}}$ that is orthogonal under a particular scalar product generalising the classical Hall scalar product on symmetric functions. As a result of this they satisfy the Cauchy identity

$$\sum_{\lambda} P_\lambda(X;q,t)Q_\lambda(Y;q,t) = \prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty} \tag{1.2.2}$$

where $Q_\lambda(X;q,t)$ is a simple multiple of the $P_\lambda(X;q,t)$. One of our main results is a higher rank Cauchy-type identity. This is contained in Theorem 3.15 and Theorem 3.16. The original Cauchy identity may be specialised to obtain the Kaneko–Macdonald q-binomial theorem for basic hypergeometric series with Macdonald polynomial argument:

$$\sum_{\lambda} t^{n(\lambda)} \frac{(a;q,t)_\lambda}{c'_\lambda(q,t)} P_\lambda(X;q,t) = \prod_{x\in X} \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

Further specialization yields the Askey–Habsieger–Kadell integral and in the limit $q \to 1^-$ the Selberg integral. In this thesis we show that a plethystically substituted version of (1.2.2) is equivalent to a q-analogue of (1.1.5) for γ a positive integer. More specifically we may claim an evaluation of the form

$$\int_{[0,1]^k} P_\mu(t;q,q^\gamma) P_\nu\left(\left[q^{\beta-\gamma}t + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\right];q,q^\gamma\right) \prod_{i=1}^k t_i^{\alpha-1}(qt_i;q)_{\beta-1} \tag{1.2.3}$$
$$\times \prod_{1\leqslant i<j\leqslant k} t_i^{2\gamma}(t_j q^{1-\gamma}/t_i;q)_{2\gamma}\, \mathrm{d}_q t$$
$$= q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} P_\mu\left[\frac{1-t^k}{1-t}\right] P_\nu\left[\frac{1-q^{\beta+(k-1)\gamma}}{1-q^\gamma}\right]$$
$$\times \prod_{i=1}^k \frac{\Gamma_q(\alpha+(k-i)\gamma+\mu_i)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(1+i\gamma)}{\Gamma_q(\alpha+\beta+(2k-\ell-i-1)\gamma+\mu_i)\Gamma_q(1+\gamma)}$$
$$\times \prod_{i=1}^k \prod_{j=1}^\ell \frac{\Gamma_q(\alpha+\beta+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma_q(\alpha+\beta+(2k-i-j)\gamma+\mu_i+\nu_j)}$$

where μ and ν are partitions satisfying $\ell(\mu) \leqslant k$ and $\ell(\nu) \leqslant \ell$ respectively. The integral (1.2.3) is a generalisation of the q-Kadell integral due to [Kan96, Proposition 5.2] and the q-Hua–Kadell integral due to Warnaar [War05, Theorem 1.4].

It should be noted that q-Selberg integrals arise through the study of other combinatorial objects. Recent work by Kim and Okada relates q-Selberg integrals to Young books, which generalise standard Young tableaux [KO17b]. This extends to work of Kim and Oh on the relationship between the standard Selberg integral and Young books [KO17a]. In another direction the work of Kim and Stanton relates q-Selberg integrals to posets and polytopes [KS17].

1.3 Knizhnik–Zamolodchikov Equations and the Mukhin–Varchenko Conjecture

The Selberg integral has also made an appearance in the representation theory of Lie algebras. This connection results from the Knizhnik–Zamolodchikov (KZ) equations. Let $\mathfrak{g}$ be a simple Lie algebra and V_λ, V_μ highest weight modules for $\mathfrak{g}$. The KZ equations take the form

$$\kappa\frac{\partial u}{\partial z} = \frac{\Omega}{z-w}u, \quad \text{and} \quad \kappa\frac{\partial u}{\partial w} = \frac{\Omega}{w-z}u.$$

Here $u(z,w)$ is a function taking values in $V_\lambda \otimes V_\mu$ and Ω is the Casimir element. One may state the KZ equations in a more general setting, and for this we refer the reader to [EFK98, §3]. In [SV91] Schechtman and Varchenko gave solutions to the KZ equations for a general simple Lie algebra $\mathfrak{g}$ in terms of hypergeometric integrals. We describe these in detail in Chapter 5. For $\mathfrak{g} = \mathfrak{sl}_2$ one specific solution is the Selberg integral. This motivated Mukhin and Varchenko to conjecture the existence of a Selberg-type integral for each simple Lie algebra $\mathfrak{g}$ [MV00].

To date the Mukhin–Varchenko conjecture has only been resolved satisfactorily in type A. The first results in this type are due to Tarasov and Varchenko [TV03] who gave several Selberg integrals for the Lie algebra $\mathfrak{sl}_3$. This integral was reproved by Warnaar in [War08] using Macdonald polynomial theory. This method also yields a more general integral formula generalising Kadell's integral (1.1.4). Warnaar managed to extend this technique to $\mathfrak{sl}_{n+1}$ through higher rank Kaneko–Macdonald-type basic hypergeometric series, of which the Kaneko–Macdonald q-binomial theorem (1.2.3) the $\mathfrak{sl}_2$ case.

In this thesis we phrase Warnaar's $\mathfrak{sl}_{n+1}$ basic hypergeometric series in terms of Cauchy-type identities. This essentially reverses the process of going from the Cauchy identity (1.2.2) to the Kaneko–Macdonald q-binomial theorem (1.2.3). Hence we are able to extend the approach in [War09] to an $\mathfrak{sl}_{n+1}$-analogue of the AFLT integral 1.1.5. This is the content of Theorem 5.4.

1.4 Outline

The content of the thesis is arranged as follows.

In Chapter 2 we review much of the basic theory of symmetric functions. This includes a section on partitions, which index families of symmetric functions. We then cover some standard definitions and facts from with minimal proofs, instead referring the reader to outside sources. The chapter concludes with a treatment of plethystic notation. This is a powerful tool in symmetric function theory which is crucial in our approach to Selberg integrals later on.

Chapter 3 concerns Macdonald polynomial theory. Following Macdonald's book [Mac15, Chapter 6] we define the Macdonald polynomials and prove a number of important properties. Of these properties the most important are the evaluation symmetry and principal specialisation formulae, which are proved in the second section of the chapter. Following this we introduce the skew Macdonald polynomials and prove several summation formulae for them. This sets up our proof of a new A_n Cauchy-type identity in the final section.

In Chapter 4 we turn to q-integrals. The first section explains the relationship between basic hypergeometric series and q-integrals. In the next section we produce a proof of the Askey–Habsieger–Kadell integral using Macdonald polynomial theory. The final section contains a proof of a new q-Selberg integral (1.2.3) which generalises the AFLT integral (1.1.5) to the q-level.

In the final chapter we use the tools developed in the previous sections to produce an A_n analogue of the AFLT integral. We motivate this by first explaining how Selberg integrals are related to simple Lie algebras. Following this we reproduce Warnaar's A_n Selberg integral and describe the complicated chain of integration. Finally we utilise the new Cauchy identity of Chapter 3 in order to produce the AFLT generalisation of Warnaar's integral. The thesis concludes with some remarks about future research directions.

CHAPTER 2

SYMMETRIC FUNCTIONS

This first chapter is devoted to the theory of symmetric functions. We begin with partitions, which index families of symmetric functions, as well as the related concept of compositions. The ring of symmetric functions is then introduced, as well as several important bases for this ring. Finally we introduce plethystic notation, a powerful tool in the theory that will prove extremely useful in subsequent chapters.

2.1 Partitions and Compositions

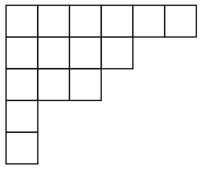
A *partition* λ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots$ such that only finitely many λ_i are nonzero. We will often write partitions as sequences, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, where we do not distinguish between the number of trailing zeroes. For instance the partitions $(2, 1)$ and $(2, 1, 0, 0)$ will be regarded as identical. We will write 0 for the unique partition of 0. We denote the sum of the entries of a partition by $|\lambda| = \sum_{i \geqslant 1} \lambda_i$. When $|\lambda| = n$ for some $n \in \mathbb{N}$ we say λ is a partition of n and write $\lambda \vdash n$. The set of all partitions will be denoted by $\mathfrak{Part}$ and the set of all partitions of a nonnegative integer n by $\mathfrak{Part}_n$. A nonzero entry in a partition is called a part. The number of parts is the length and written $\ell(\lambda)$. As an example consider $\lambda = (6, 4, 3, 1, 1)$. Then $|\lambda| = 15$ and $\ell(\lambda) = 5$. An important partition is the *staircase partition* denoted $\delta = \delta^{(n)} = (n-1, n-2, \dots, 1, 0)$. This is a partition of $\binom{n}{2}$ and has length $\ell(\delta) = n - 1$.

Relaxing the condition that the entries of a partition occur in weakly decreasing order leads to the concept of a *weak composition*. This is a finite sequence of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. The integer n is called the length and we write $\ell(\alpha) = n$. As for partitions we say α is a weak composition of k if $|\alpha| = \sum_{i=1}^{n} \alpha_i = k$. Given any (weak) composition it is possible to rearrange the entries such that they are in weakly decreasing order. Hence one may speak of the unique partition in the $\mathfrak{S}_n$ orbit of the composition α.

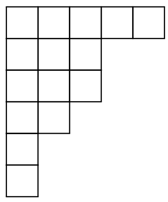
It will also be convenient to write partitions in a different manner. Given a partition λ we define the multiplicity of a part to be the number of times that integer occurs as a part of λ. This will be denoted by $m_i(\lambda)$ or simply m_i when λ is understood. An alternative notation for partitions is then $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots)$. It is customary in this notation to omit parts that do not occur, and to omit the exponent when $m_i(\lambda) = 1$. For example setting $\lambda = (6, 4, 3, 1, 1)$ as before we see that $\lambda = (1^2 346)$.

Another way of representing partitions is through a graphical method. The

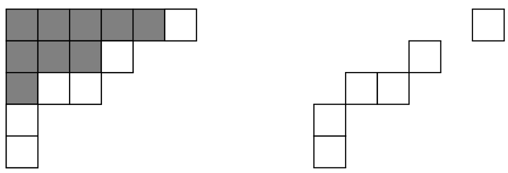
Young diagram for a partition λ is the left-justified array of n squares in the plane with λ_i squares placed on the ith row. Taking $\lambda = (6, 4, 3, 1, 1)$ as before we have



It should be noted that this is the English convention for description of partitions. In the French convention the Young diagram is left-justified with the parts decreasing upward. The Young diagram allows us to define the notion of a *conjugate partition.* Given a partition λ the conjugate partition λ' is defined by reflecting the squares along the main diagonal. Hence the partition conjugate to $\lambda = (6, 4, 3, 1, 1)$ is $\lambda' = (5, 3, 3, 2, 1, 1)$, and graphically is

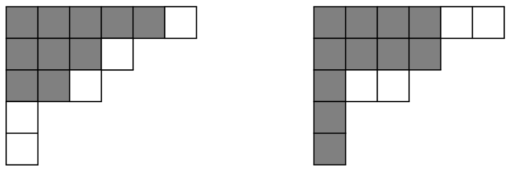


From this definition we see that $\ell(\lambda) = \lambda'_1$. Observe that the number of squares in the first column counts the number of parts of the partition that are at least 1. Similarly the number of squares in the ith column counts the number of parts that are at least i. Hence we may also describe each part in the conjugate partition as $\lambda'_i = |\{\lambda_i : \lambda_i \geqslant i\}|$. It is also possible to express the multiplicity of a part in terms of the conjugate partition. Indeed we have $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$. We say that a partition μ is *contained in* λ, written $\mu \subseteq \lambda$, if the diagram of μ is contained in the diagram of λ. Containment is a partial order on $\mathfrak{Part}$, and the resulting poset is called Young's lattice. Whenever $\mu \subseteq \lambda$ we may form the skew diagram λ/μ by removing the squares of μ from those of λ. The partition $\mu = (5, 3, 1)$ is contained in $\lambda = (6, 4, 3, 1, 1)$ and has the following skew shape



When μ is the empty partition it is clear that $\lambda/0 = \lambda$. A skew diagram λ/μ is said to be a *horizontal strip* if it contains at most one square in every column. Similarly a *vertical strip* contains at most one square in every row. Our example above is neither a vertical nor a horizontal strip. However if we set $\mu = (5, 3, 2)$ and $\nu = (4, 4, 1, 1, 1)$ then λ/μ is only a vertical strip and λ/ν

is only a horizontal strip



where the partition we are removing is shaded grey. A *horizontal* or *vertical* *r-strip* is a strip with containing r boxes. A subset of a skew diagram is called *connected* if each square shares at least one edge with another square in the diagram.

Let X be a set of combinatorial objects. A *statistic* on X is a combinatorial rule that assigns each element of X a natural number $n \in \mathbb{N} =: \{0, 1, 2, \ldots\}$. For any partition λ, the statistic $n(\lambda)$ is defined by

$$n(\lambda) = \sum_{i \geqslant 1} (i-1)\lambda_i. \tag{2.1.1}$$

For example computing $n(\lambda)$ in the case $\lambda = (6, 4, 3, 1, 1)$ yields $n(\lambda) = 17$. One may interpret this is a different manner. In (2.1.1) we are essentially giving row i of the Young diagram of λ the weight $i - 1$. Let us populate the rows in this fashion, giving for our example

0	0	0	0	0	0
1	1	1	1		
2	2	2			
3					
4					

So the statistic $n(\lambda)$ can be seen as the sum over the squares in the Young diagram with the given weights. However summing over the columns tells us that

$$n(\lambda) = \sum_{i \geqslant 1} \binom{\lambda_i'}{2}. \tag{2.1.2}$$

There are several commonly used orderings on partitions, however we will only discuss the dominance ordering in detail. For an explanation of the lexicographic ordering, another standard ordering, see [Mac15, p. 6] or [Loe11, p. 389].

Definition 2.1. Let λ, μ be partitions such that $|\lambda| = |\mu|$. We say that μ is *dominated* by λ (or μ is less than λ in the dominance ordering) if for each $k \geqslant 1$ it holds that

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i.$$

This will be written $\mu \leqslant \lambda$ with the notation $\mu < \lambda$ reserved for when $\lambda \neq \mu$.

The dominance ordering is in general a partial order on $\mathfrak{Part}_n$. However if $n \leqslant 5$ then it is a total ordering. For $n = 6$ this fails as the partitions $(3, 1, 1, 1)$ and $(2, 2, 2)$ are not comparable. We now prove a property of partitions and their conjugates under the dominance ordering.

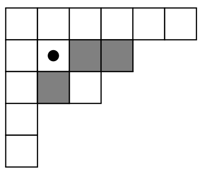
Proposition 2.2. *Let λ and μ be partitions such that $|\lambda| = |\mu|$. Then $\lambda \geqslant \mu$ if and only if $\mu' \geqslant \lambda'$.*

Proof. See [Mac15, p. 7]. ∎

The final concept from partition theory we require are the notions of arms and legs. We may think of the diagram of a partition λ as the set of points $(i, j) \in \mathbb{N}^2$ such that $i \leqslant \ell(\lambda)$ and $j \leqslant \lambda_i$ for each i. The indexing is as in a matrix, increasing from right to left and top to bottom. Let $s \in \lambda$ be a square in the diagram of λ. Define the *arm length* and *leg length* by

$$a(s) = \lambda_i - j, \qquad \text{and} \qquad l(s) = \lambda'_j - i \tag{2.1.3}$$

respectively. These are equivalently defined to be the number of squares directly to the right of s and the number of squares directly below s respectively. For the partition $\lambda = (6, 4, 3, 1, 1)$ the square $s = (2, 2) \in \lambda$ has $a(s) = 2$ and $l(s) = 1$. Visually this can be represented as



with the bulleted square being $(2, 2)$ with its arm and leg shaded. The *hook length* of a square s is $h(s) = a(s) + l(s) + 1$. One also has the *arm colength* and *leg colength*

$$a'(s) = j - 1 \qquad \text{and} \qquad l'(s) = i - 1. \tag{2.1.4}$$

Again these are equivalently the number of squares immediately to the left of s and the number of squares immediately above s. This leads to another definition of $n(\lambda)$ from (2.1.1). For each positive integer $i \leqslant \ell(\lambda)$ there are precisely λ_i squares with leg colength $(i - 1)$. Hence we obtain

$$n(\lambda) = \sum_{s \in \lambda} l'(s). \tag{2.1.5}$$

2.2 The Ring of Symmetric Functions

In this section we introduce the ring of symmetric functions. Our treatment covers only the most essential facts, with many proof omitted. For a comprehensive treatment of the theory see [Mac15, §1] and [Sta99, §7]. Fix a positive integer n. We define an *alphabet of cardinality* n to be a set of n commuting indeterminates. Throughout this section we denote such an alphabet by $X = (x_1, x_2, \ldots, x_n)$. A polynomial $f \in \mathbb{Z}[X]$ is called *symmetric* if it is invariant under the natural action of the symmetric group on the indices, i.e., for any $w \in \mathfrak{S}_n$ it holds that

$$f(x_1, \ldots, x_n) = f(x_{w(1)}, \ldots, x_{w(n)}).$$

The set of all symmetric polynomials forms a subring of $\mathbb{Z}[X]$, denoted Λ_n. One may also consider the subring of symmetric polynomials homogeneous of degree k which is written Λ_n^k. This allows us to express the ring of symmetric polynomials as a graded ring,

$$\Lambda_n = \bigoplus_{k \geqslant 0} \Lambda_n^k. \tag{2.2.1}$$

To see that this is a valid grading let $i, j \geqslant 0$ be nonnegative integers and $f \in \Lambda_n^i, g \in \Lambda_n^j$. Then the product fg is clearly homogeneous of degree $i+j$ and hence $fg \in \Lambda_n^{i+j}$. We will now meet our first important class of symmetric functions.

Definition 2.3. Let X be an alphabet of cardinality n and λ a partition. The *monomial symmetric polynomial* indexed by the partition λ is defined by

$$m_\lambda(X) = \sum_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where the sum is over all distinct permutations α of λ. If $\ell(\lambda) > n$ then $m_\lambda(X)$ is defined to be zero.

As an example let $X = (x_1, x_2, x_3)$ and $\lambda = (2, 1)$. Then

$$m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2.$$

The monomial symmetric polynomials form a basis for Λ_n. To see this consider $f \in \Lambda_n^k$. Then for some coefficients c_α we may write

$$f(X) = \sum_\alpha c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where the sum is over all weak compositions α such that $|\alpha| = k$ and $\ell(\lambda) \leqslant n$. We know that each α occurs in the $\mathfrak{S}_n$-orbit of some unique partition. It follows that

$$f(X) = \sum_\lambda c_\lambda m_\lambda(X),$$

where the sum is over all partitions λ such that $\lambda \vdash k$ and $\ell(\lambda) \leqslant n$. Hence the set $\{m_\lambda(X) : \lambda \vdash k, \ell(\lambda) \leqslant n\}$ forms a basis for Λ_n^k. This shows that the dimension of Λ_n^k is the number of partitions of k with length at most n. If

n is greater than k then the dimension is simply the number of partitions of k. Furthermore the grading (2.2.1) tells us that $\{m_\lambda(X) : \ell(\lambda) \leqslant n\}$ forms a basis for Λ_n.

When working with symmetric polynomials the number of variables is often irrelevant, only requiring that the number n is sufficiently large so that vanishing does not occur. This idea may be formalised by introducing the ring of symmetric functions in countably many variables $X = (x_1, x_2, x_3, \dots)$. When dealing with this new object we drop the subscript and simply write Λ. The construction proceeds as follows. Let $m \geqslant n$ be positive integers and consider the homomorphism

$$\rho_{n,m} : \Lambda_m \to \Lambda_n \tag{2.2.2}$$

defined by sending x_i to x_i for $1 \leqslant i \leqslant n$ and the remaining $m - n$ variables to zero. These homomorphisms act on the basis of monomial symmetric polynomials by

$$\rho_{n,m}(m_\lambda(x_1, \dots, x_m)) = m_\lambda(x_1, \dots, x_n)$$

if $\ell(\lambda) \leqslant n$ and $\rho_{n,m}(m_\lambda) = 0$ otherwise. This shows that $\rho_{n,m}$ is a surjection for all $m \geqslant n$. Furthermore the restriction

$$\rho^k_{n,m} : \Lambda^k_m \to \Lambda^k_n \tag{2.2.3}$$

is also a surjection. It is clear that when $m = n$ the map $\rho^k_{n,n}$ is the identity and when $l \geqslant m \geqslant n$ we have that the following diagram commutes:

$$\begin{array}{ccc} \Lambda^k_l & \xrightarrow{\rho_{m,l}} & \Lambda^k_m \\ & \searrow^{\rho_{n,l}} & \downarrow{\scriptstyle \rho_{n,m}} \\ & & \Lambda^k_n \end{array}$$

The collection of $\mathbb{Z}$-modules Λ^k_n for all n and fixed k together with the homomorphisms $\rho^k_{n,m}$ form an inverse (or directed) system. Hence the ring of *symmetric functions homogeneous of degree* k in arbitrary many variables is defined by

$$\Lambda^k := \varprojlim_n \Lambda^k_n, \tag{2.2.4}$$

which is the inverse (or direct) limit of the $\mathbb{Z}$-modules Λ^k_n with respect to the $\rho^k_{n,m}$. The *ring of symmetric functions* is therefore

$$\Lambda := \bigoplus_{k \geqslant 0} \Lambda^k. \tag{2.2.5}$$

Elements and identities in the ring Λ may be reduced to Λ_n by the natural projection maps $\pi_n : \Lambda \to \Lambda_n$ which extracts the nth element of the sequence corresponding to $f \in \Lambda$. Here we have only defined the ring of symmetric functions over $\mathbb{Z}$. In later sections we will use the ring of symmetric functions over both $\mathbb{Q}$ and $\mathbb{Q}(q,t)$. We define these by the tensor product

$$\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{F}. \tag{2.2.6}$$

In both cases the monomial symmetric functions still form a basis.

Remark. The elements of Λ are no longer polynomials in the variables $X = (x_1, x_2, x_3, \dots)$ but rather infinite formal sums of monomials. Hence as in literature we refer to these elements as *symmetric functions.* While being formal sums of actual monomials, elements of Λ are finite sums of the monomial symmetric functions.

We will now introduce several classical classes of symmetric functions. The first of these are the complete symmetric functions.

Definition 2.4. For $r \geqslant 0$ an integer the *complete symmetric function* indexed by r is

$$h_r(X) = \sum_{\lambda \vdash r} m_\lambda(X).$$

So the complete symmetric function $h_r(X)$ is the sum of all monomials of degree r in the variables X. This is extended to partitions by

$$h_\lambda(X) = \prod_{i \geqslant 1} h_{\lambda_i}(X).$$

It is immediate that $h_0(X) = 1$. For an example let $X = (x_1, x_2, x_3)$. Then

$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3,$$

and $h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Therefore

$$h_{(2,1)}(x_1, x_2, x_3) = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3).$$

In terms of the monomial symmetric functions the above is equivalent to

$$h_{(2,1)}(x_1, x_2, x_3) = m_{(1)}(x_1, x_2, x_3)\big(m_{(2)}(x_1, x_2, x_3) + m_{(1,1)}(x_1, x_2, x_3)\big).$$

The complete symmetric functions admit a simple generating function. For t another indeterminate we have

$$\sigma_t(X) := \sum_{r=0}^{\infty} h_r(X) t^r = \prod_{i \geqslant 1} \frac{1}{1 - x_i t}, \tag{2.2.7}$$

where we have used the notation of [Las03, p. 5]. When $t = 1$ the subscript on the left is omitted and we simply write $\sigma(X)$. This formal power series in t has a number of important applications when using plethystic notation, which is explained in Section 2.3. The polynomials h_r for $r \geqslant 0$ are in fact algebraically independent over $\mathbb{Z}$ and generate Λ [Sta99, Corollary 7.6.2]. Therefore we have

$$\Lambda \cong \mathbb{Z}[h_1, h_2, h_3, \dots].$$

Furthermore if we index h_λ over all partitions then they form a basis for Λ. The next important class of symmetric functions are the elementary symmetric functions.

Definition 2.5. For an integer $r \geqslant 0$ the *elementary symmetric polynomial* indexed by r is

$$e_r(X) = m_{(1^r)}(X),$$

so that $e_r(x)$ is the sum over all products of r distinct variables x_i. This is similarly extended to partitions by

$$e_\lambda(X) = \prod_{i \geqslant 1} e_{\lambda_i}(X).$$

By definition it follows that $e_0(X) = 1$. As an example if we consider the alphabet $X = (x_1, x_2, x_3)$ so that

$$e_2(X) = x_1x_2 + x_1x_3 + x_2x_3.$$

Also $e_1(X) = h_1(x)$. Therefore

$$e_{(2,1)}(X) = (x_1 + x_2 + x_3)(x_1x_2 + x_1x_3 + x_2x_3).$$

As in the complete case, the elementary symmetric functions admit a simple generating function. Again with the notation of [Las03, p. 5], for t an indeterminate

$$\lambda_t(X) := \sum_{r=0}^{\infty} e_r(X)t^r = \prod_{i \geqslant 1}(1 + x_i t). \tag{2.2.8}$$

Therefore by (2.2.7) and (2.2.8) one has

$$\sigma_t(X)\lambda_{-t}(x) = 1. \tag{2.2.9}$$

This is the first interesting interaction between the elementary and complete symmetric polynomials. To state another we note that the e_r are algebraically independent over $\mathbb{Z}$ and generate Λ [Sta99, Theorem 7.4.4]. Hence we have

$$\Lambda \cong \mathbb{Z}[e_1, e_2, e_3, \dots].$$

Indexing the elementary symmetric functions over all partitions gives another basis for Λ. We now introduce an involution of Λ that interchanges the complete and elementary symmetric functions. Define $\omega : \Lambda \to \Lambda$ by

$$\omega(e_r) = h_r. \tag{2.2.10}$$

As ω is also an algebra homomorphism it follows that $\omega(e_\lambda) = h_\lambda$ for any partition λ. To see that this is indeed an involution extract the coefficient of t^n on both sides of (2.2.9) to obtain

$$0 = \sum_{i=0}^{n} (-1)^i e_i h_{n-i}$$

for all $n \geqslant 1$. Applying ω to this expression gives

$$0 = \sum_{i=0}^{n} (-1)^i h_i \omega(h_{n-i}) = (-1)^n \sum_{i=0}^{n} (-1)^i \omega(h_i) h_{n-i}.$$

Therefore we have that

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = \sum_{i=0}^{n} (-1)^i \omega(h_i) h_{n-i}.$$

This implies that $\omega(h_i) = e_i$, and hence ω^2 must be the identity.

We now turn to another class of symmetric functions, the power sum symmetric polynomials.

Definition 2.6. For $r \geqslant 1$ an integer the *power sum symmetric polynomial* indexed by r is defined by

$$p_r(X) = \sum_{i \geqslant 1} x_i^r.$$

As before this is extended to partitions λ by

$$p_\lambda(X) = \prod_{i \geqslant 1} p_{\lambda_i}(X).$$

As an example we shall compute $p_{(2,1)}(x_1, x_2, x_3)$ and compare this with $h_{(2,1)}(x_1, x_2, x_3)$. It is clear that $p_1(x) = h_1(x)$ for an arbitrary alphabet x and thus $p_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Finally

$$p_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

and so

$$p_{(2,1)}(x_1, x_2, x_3) = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2).$$

In terms of the monomial symmetric functions the power sums are

$$p_r(X) = m_{(r)}(X).$$

Unlike the families we have seen before, the power sums do not form another $\mathbb{Z}$-basis for Λ. Instead one has that the p_r are algebraically independent over $\mathbb{Q}$ and generate $\Lambda_\mathbb{Q}$ as a $\mathbb{Q}$-algebra [Sta99, Corollary 7.7.2]. Therefore we have

$$\Lambda_\mathbb{Q} \cong \mathbb{Q}[p_1, p_2, p_3, \ldots]$$

Under the action of ω the power sums satisfy [Mac15, p. 24]

$$\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda.$$

The power sums admit a generating function as with the complete and elementary symmetric functions, however it is not as simple. It is given by

$$\Psi_t(X) = \sum_{r \geqslant 1} p_r(X) t^{r-1}. \tag{2.2.11}$$

We may write this in terms of $\sigma_t(x)$. Indeed observe that

$$\begin{aligned}
\Psi_t(X) &= \sum_{r \geqslant 1} \sum_{i \geqslant 1} x_i^r t^{r-1} \\
&= \sum_{i \geqslant 1} x_i \sum_{r \geqslant 1} x_i^{r-1} t^{r-1} \\
&= \sum_{i \geqslant 1} \frac{x_i}{1 - x_i t} \\
&= \sum_{r \geqslant 1} \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{1}{1 - x_i t}.
\end{aligned}$$

By linearity of the formal derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ and the properties of the natural logarithm we see that

$$\Psi_t(X) = \frac{\mathrm{d}}{\mathrm{d}t} \log \prod_{i \geqslant 1} \frac{1}{1 - x_i t} = \frac{\mathrm{d}}{\mathrm{d}t} \log \sigma_t(X). \tag{2.2.12}$$

Setting $t \mapsto -t$ gives a similar result for $\lambda_t(X)$.

With these important classes of symmetric functions established we may now define the *Hall scalar product* on symmetric functions. This requires that the complete and monomial symmetric functions are orthonormal:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}, \tag{2.2.13}$$

where $\delta_{\lambda\mu}$ is the Kronecker delta. The power sum symmetric polynomials are orthogonal under this inner product [Sta99, Proposition 7.9.3]

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}, \tag{2.2.14}$$

where the quantity z_λ is known to be

$$z_\lambda = \prod_{i \geqslant 1} m_i(\lambda)! \cdot i^{m_i(\lambda)}. \tag{2.2.15}$$

One may interpret z_λ in terms of the symmetric group: it is the size of the centraliser of any element of cycle type λ in $\mathfrak{S}_n$ [Sag01, p. 3]. It may also be used to express the generating functions $\sigma_z(x)$ (2.2.7) and $\lambda_z(x)$ (2.2.8) in terms of the power sums.

Proposition 2.7. *For t an indeterminate one has*

$$\sigma_t(X) = \sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) t^{|\lambda|} \tag{2.2.16a}$$

$$\lambda_t(X) = \sum_\lambda \frac{(-1)^{|\lambda|-\ell(\lambda)}}{z_\lambda} p_\lambda(X) t^{|\lambda|}. \tag{2.2.16b}$$

Proof. We first prove (2.2.16a). By (2.2.12) there holds

$$\sigma_t(X) = \exp\Bigg(\sum_{r\geqslant 1} p_r(X)\frac{t^r}{r}\Bigg).$$

Using the properties of the exponential and its power series expansion we obtain

$$\sigma_t(X) = \prod_{r\geqslant 1} \sum_{m_r=0}^{\infty} \frac{(p_r(X)t^r)^{m_r}}{r^{m_r}\cdot m_r!}.$$

The combined product over r and corresponding sum over m_r may be interpreted as a sum over all partitions. Here, the m_r are treated as the multiplicities of r, and the product ensures each positive integer takes each possible value for its multiplicity. Given any finite sequence of multiplicities (in lexicographic order) there exists a unique partition corresponding to this sequence. Hence this becomes a sum over partitions and we obtain

$$\sigma_t(X) = \sum_\lambda \frac{1}{z_\lambda} p_r(X) t^{|\lambda|},$$

as desired. The identity (2.2.16b) follows by applying the involution ω. ∎

By extracting the coefficient of t^n on both sides of (2.2.16) we obtain the identities

$$h_n(X) = \sum_{\lambda\vdash n} \frac{1}{z_\lambda} p_\lambda(X),$$

$$e_n(X) = \sum_{\lambda\vdash n} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} p_\lambda(X).$$

Duality of two bases under the scalar product (2.2.13) is in general equivalent to Cauchy-type identities for symmetric functions. By a Cauchy-type identity we mean a sum over symmetric functions on different alphabets which may equivalently be expressed as a product over the letters in said alphabets.

Proposition 2.8. *Let X and Y be countable alphabets. Then*

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) p_\lambda(Y) = \prod_{i,j=1}^{\infty} \frac{1}{1-x_i y_j} \tag{2.2.17}$$

where the sum is over all partitions λ.

Remark. The proof of the above statement will become very simple when we introduce plethystic notation in Section 2.3. Therefore we assume the result for now. For a more complicated proof that avoids plethystic notation see [Sta99, Proposition 7.7.4].

Using this identity, and the fact that the power sums form a $\mathbb{Q}$-basis for Λ we may prove the following important result.

Proposition 2.9. *For each nonnegative integer n let $\{u_\lambda\}, \{v_\mu\}$ be $\mathbb{Q}$-bases for $\Lambda_{\mathbb{Q},n}$. Then the following are equivalent:*

1. *For all partitions λ, μ*
$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}. \tag{2.2.18}$$

2. *For alphabets X and Y one has*
$$\sum_\lambda u_\lambda(X) v_\lambda(Y) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}. \tag{2.2.19}$$

Proof. Define the rescaling of the power sums by $\tilde{p}_\lambda(X) = p_\lambda(X)/z_\lambda$. These form a basis for $\Lambda_{\mathbb{Q}}$ that is orthonormal to the basis $\{p_\lambda\}$. Let $b_{\lambda\nu}, c_{\mu\omega} \in \mathbb{F}$ be the coefficients of the bases $\{u_\lambda\}$ and $\{v_\lambda\}$ when expanded as follows

$$u_\lambda = \sum_\nu b_{\lambda\nu} \tilde{p}_\nu \quad \text{and} \quad v_\mu = \sum_\omega c_{\mu\omega} p_\omega.$$

Then by linearity and the fact that $\langle \tilde{p}_\lambda, p_\mu \rangle = \delta_{\lambda\mu}$ the equation (2.2.18) is equivalent to
$$\sum_\nu b_{\lambda\nu} c_{\mu\nu} = \delta_{\lambda\mu}. \tag{2.2.20}$$

However by Proposition 2.8 we may write (2.2.19) as

$$\sum_\lambda u_\lambda(X) v_\lambda(Y) = \sum_\omega \tilde{p}_\omega(X) p_\omega(Y). \tag{2.2.21}$$

Expanding $u_\lambda(X)$ and $v_\lambda(Y)$ as above then equating coefficients tells us that that (2.2.21) is equivalent to

$$\sum_\lambda b_{\lambda\nu} c_{\lambda\omega} = \delta_{\nu\omega}. \tag{2.2.22}$$

Therefore as (2.2.20) is equivalent to (2.2.22) we have that (2.2.18) is equivalent to (2.2.19). This proves the result. ∎

2.3 Plethystic Notation

Plethystic notation (or λ-ring notation) is a powerful tool in symmetric function theory that allows for the manipulation of the alphabet the ring of symmetric functions is defined over. It further allows many familiar properties of arithmetic to be extended to the level of alphabets. In the previous section it was explained that the power sum symmetric functions p_r are algebraically independent over $\mathbb{Q}$ and generate the ring of symmetric functions $\Lambda_{\mathbb{Q}}$. Let $\mathbb{F}$ be any field containing $\mathbb{Q}$ and A be an algebra over $\mathbb{F}$. Then assigning the symmetric functions p_r values in A determines a map that may be extended uniquely to an $\mathbb{F}$-algebra homomorphism $\Lambda_{\mathbb{Q}} \to A$. It is this property that underlies the theory of plethystic substitution. For more complete accounts refer to [Hag08, Las03]

Let E be a formal series of rational functions in the indeterminates a_i. Denote by $p_r[E]$ the result of replacing a_i by a_i^r in E. Extending this to an arbitrary $f \in \Lambda_{\mathbb{F}}$, the expression $f[E]$ is called the *plethystic substitution* of E into f. Here $f[E]$ is the image of f under the homomorphism sending p_r to $p_r[E]$. Here the square brackets are known as the plethystic brackets, which flag when a plethystic substitution is being used.

There are a number of illustrative examples of plethystic substitution. Here we will define some of these relating to the power sums p_r and generating function $\sigma_1(x) := \sigma(x)$ which is (2.2.7) with $t = 1$. In this case

$$\sigma[X] = \prod_{x \in X} \frac{1}{1-x} = \exp\Big(\sum_{k \geqslant 1} p_k[X]/k\Big) \tag{2.3.1}$$

which follows readily from (2.2.12).

Let X and Y be arbitrary countable alphabets. We consider X and Y as sets rather than ordered tuples, as the order of variables is irrelevant when working with symmetric functions. The sum $X+Y$ is defined to be the disjoint union of these sets. By this convention, inside of the plethystic brackets the alphabet $\{x_1, x_2, \dots\}$ is thought of as the formal sum $x_1 + x_2 + \cdots$. Hence any symmetric function f satisfies

$$f[X] = f(X).$$

On the power sums and $\sigma(X)$ we have

$$p_r[X+Y] = p_r[X] + p_r[Y] \tag{2.3.2a}$$

$$\sigma[X+Y] = \sigma[X]\sigma[Y] \tag{2.3.2b}$$

Here (2.3.2a) is easily established and (2.3.2b) follows from (2.3.2a) together with (2.3.1). The product of alphabets is the Cartesian product of the sets X and Y denoted by juxtaposition, i.e.,

$$XY := \{xy : x \in X, y \in Y\}, \tag{2.3.3}$$

where we assume the letters commute. This implies that $XY = YX$. On the power sums we have

$$p_r[XY] = p_r[X]p_r[Y]. \tag{2.3.4}$$

With this notation the alphabet aX for a parameter a is interpreted as the product of the single letter alphabet $\{a\}$ with X. If $f \in \Lambda_{\mathbb{F}}$ is homogeneous of degree k then

$$f[aX] = a^k f[X]. \tag{2.3.5}$$

It is important to note that although a is a free parameter, it may not take on specific values. The subtraction of the alphabets X and Y takes the following form on the power sums:

$$p_r[X - Y] = p_r[X] - p_r[Y]. \tag{2.3.6}$$

In general we write $-Y$ for the alphabet $\{\}-Y$ where $\{\}$ is the empty alphabet with no letters and not for the product $\{-1\}Y$. So by this convention

$$p_r[-Y] = -p_r[Y],$$

which is not equal to $(-1)^r p_r[Y]$ which is the result of replacing each letter in Y by its negative. Of course this is still a valid plethystic substitution and is represented by ϵ inside the plethystic brackets. Therefore if f is homogeneous of degree k we have

$$f[\epsilon X] = (-1)^k f[X].$$

In general there is no way to define the ratio of alphabets $f[X/Y]$. However in the case that Y is a single letter alphabet then we may simply treat its single letter y as the letter y^{-1} in the ratio so that the product is well defined. It will often be convenient to replace alphabets for symmetric functions by geometric progressions involving individual variables and parameters. For example inside of plethystic brackets it is understood that

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots. \tag{2.3.7}$$

We then see that

$$\frac{X}{1-t} = X(1 + t + t^2 + t^3 + \dots). \tag{2.3.8}$$

We will now consider $\sigma[XY]$ for arbitrary alphabets X and Y. It is clear that

$$\sigma[XY] = \prod_{x \in X} \prod_{y \in Y} \frac{1}{1 - xy}. \tag{2.3.9}$$

With this established we now return to the proof of Proposition 2.8.

Proof of Proposition 2.8. We wish to show that

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) p_\lambda(Y) = \prod_{x\in X}\prod_{y\in Y} \frac{1}{1-xy}.$$

To see this make the substitution $X \mapsto XY$ in (2.2.16a) and set $t = 1$ to obtain

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(XY) = \sigma[XY].$$

Applying (2.3.4) and (2.3.9) yields the result. ∎

By virtue of (2.3.9) and Proposition 2.9 the duality between bases two $\mathbb{Q}$-bases for Λ, $\{u_\lambda\}$ and $\{v_\lambda\}$, may be expressed as the simple identity

$$\sum_\lambda u_\lambda[X] v_\lambda[Y] = \sigma[XY].$$

One may also phrase the action of the involution ω (2.2.10) plethystically. Indeed the plethystic substitution $X \mapsto -\epsilon X$ acts in the following way on the power sums

$$\begin{aligned} p_\lambda[-\epsilon X] &= \prod_{i\geqslant 1} p_{\lambda_i}[-\epsilon X] \\ &= \prod_{i\geqslant 1} (-1)^{\lambda_i - 1} p_{\lambda_i}[X] \\ &= (-1)^{|\lambda| - \ell(\lambda)} p_\lambda[X]. \end{aligned}$$

This is precisely the action of ω.

We will now prove a formula for $\sigma(X)$ under the plethystic substitution $X \mapsto \frac{a-b}{1-t}X$. Define the infinite q-shifted factorial, here viewed as a formal power series in parameters a and q, by

$$(a;q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j). \tag{2.3.10}$$

Lemma 2.10. *Let a, b, and q be indeterminates. Then*

$$\sigma\left[\frac{a-b}{1-q}\right] = \frac{(b;q)_\infty}{(a;q)_\infty}. \tag{2.3.11}$$

Proof. We begin by replacing $1/(1-q)$ with its geometric series to obtain

$$\sigma\left[\frac{a-b}{1-q}\right] = \sigma[(a-b)(1 + q + q^2 + \dots)].$$

Now by (2.3.2b) we have

$$\sigma\left[\frac{a-b}{1-q}\right] = \prod_{i\geqslant 0} \sigma[aq^i - bq^i] = \prod_{i\geqslant 0} \frac{\sigma[aq^i]}{\sigma[bq^i]}.$$

Here we have used that $\sigma[-X] = 1/\sigma[X]$ which is a simple consequence of (2.3.1). Applying the definition of $\sigma[X]$ shows us that

$$\prod_{i\geqslant 0} \frac{\sigma[aq^i]}{\sigma[bq^i]} = \prod_{i\geqslant 0} \frac{1-bq^i}{1-aq^i} = \frac{(b;q)_\infty}{(a;q)_\infty},$$

as desired. ■

We conclude this section with a discussion of plethystic notation and limits. Frequently in the sequel we will take limits involving symmetric functions on which the alphabet is dependent. Let $f \in \Lambda$ be a symmetric function and consider

$$f\left[\frac{1-q^\gamma}{1-q}\right].$$

This is a symmetric function defined on the difference of the alphabets $1+q+q^2+\cdots$ and $q^\gamma(1+q+q^2+\cdots)$. Upon taking the formal limit as $q \to 1$ it is well-known that

$$\lim_{q\to 1} \frac{1-q^\gamma}{1-q} = \gamma.$$

Hence inside of the plethystic brackets we understand that the notation

$$f[\gamma] := \lim_{q\to 1} f\left[\frac{1-q^\gamma}{1-q}\right].$$

If γ is a positive integer then this is simply the specialised symmetric function

$$f(\underbrace{1,1,\ldots,1}_{\gamma \text{ times}}).$$

However if γ is not an integer then we take the above limit as the definition of $f[\gamma]$. This will prove an important distinction when turning sums into integrals in Chapters 4 and 5.

CHAPTER 3

MACDONALD POLYNOMIALS

The Macdonald polynomials are a celebrated class of symmetric functions that unify much of the classical theory whilst satisfying many powerful and interesting properties themselves. Hinted at by the work of Kadell, their original definition by Macdonald is somewhat obscure and hence we outline only the most important details here. Later on we prove a new generalised Cauchy identity for Macdonald polynomials.

3.1 Definition and Properties

Throughout this section let $\mathbb{F} = \mathbb{Q}(q,t)$ be the field of rational functions in the independent parameters q and t. As per (2.2.6) we may define the ring of symmetric functions over this field by

$$\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(q,t). \tag{3.1.1}$$

We will consider many q,t-analogues of the results from classical symmetric function theory. The analogue of the quantity z_λ (2.2.15) is

$$z_\lambda(q,t) = z_\lambda \prod_{i\geqslant 1} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{3.1.2}$$

One may define a q,t-deformation of the Hall scalar product on Λ by demanding that the polynomials p_λ are orthogonal, i.e.,

$$\langle p_\lambda, p_\mu\rangle := \langle p_\lambda, p_\mu\rangle_{q,t} = z_\lambda(q,t)\delta_{\lambda\mu}. \tag{3.1.3}$$

Unless otherwise stated we will now use the q,t-Hall scalar product whenever taking the inner product of symmetric functions. Analogous to the classical case, any two bases for $\Lambda_{\mathbb{F}}$ that are orthonormal under the q,t-Hall scalar product satisfy a Cauchy identity. For arbitrary alphabets X, Y the q,t-analogue of the Cauchy kernel has the form

$$\prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty}. \tag{3.1.4}$$

An application of Lemma 2.10 with the plethystic substitution $(a-b)/(1-q) \mapsto XY(1-t)/(1-q)$ with $a=1$ and $b=t$ allows us to write the q,t-Cauchy kernel as

$$\sigma\Big[XY\frac{1-t}{1-q}\Big] = \prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty}. \tag{3.1.5}$$

As before we first prove a Cauchy identity for the power sums directly.

Proposition 3.1. *The power sum symmetric functions satisfy*

$$\sum_\lambda \frac{1}{z_\lambda(q,t)} p_\lambda(X) p_\lambda(Y) = \sigma\left[XY\frac{1-t}{1-q}\right] \tag{3.1.6}$$

where the sum is over all partitions λ.

Proof. We restate Proposition 2.8 plethystically, so that

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) p_\lambda(Y) = \sigma[XY] = \prod_{x\in X}\prod_{y\in Y}\frac{1}{1-xy}.$$

If we carry out the plethystic substitution $XY \mapsto XY(1-t)/(1-q)$ we see that

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) p_\lambda\left[Y\frac{1-t}{1-q}\right] = \sigma\left[XY\frac{1-t}{1-q}\right].$$

Now note that by the homogeneity of the power sums along with (2.3.4) and (2.3.6) we have

$$p_\lambda\left[Y\frac{1-t}{1-q}\right] = p_\lambda(Y)\prod_{i\geqslant 1}\frac{1-t^{\lambda_i}}{1-q^{\lambda_i}}.$$

Using this identity we see that

$$\begin{aligned}\sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) p_\lambda\left[Y\frac{1-t}{1-q}\right] &= \sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) p_\lambda(Y)\prod_{i\geqslant 1}\frac{1-t^{\lambda_i}}{1-q^{\lambda_i}}\\ &= \sum_\lambda \frac{1}{z_\lambda(q,t)} p_\lambda(X) p_\lambda(Y).\end{aligned}$$

The last equality follows by the definition of $z_\lambda(q,t)$ from (3.1.2). ■

The statement of Proposition 3.1 may be generalised in the following way.

Proposition 3.2. *For each nonnegative integer n let $\{u_\lambda\}$ and $\{v_\lambda\}$ be $\mathbb{F}$-bases for $\Lambda_{\mathbb{F}}$. Then the following are equivalent:*

1. *For all λ, μ there holds*

$$\langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu}. \tag{3.1.7}$$

2. *The Cauchy identity holds, so*

$$\sum_\lambda u_\lambda(X) v_\lambda(Y) = \prod_{x\in X}\prod_{y\in Y}\frac{(txy;q)_\infty}{(xy;q)_\infty}. \tag{3.1.8}$$

Proof. The proof is identical to that of Proposition 2.9. ■

With this established we may now state the existence theorem for the Macdonald polynomials. We leave this without proof. The original proof of Macdonald constructs the Macdonald polynomials as eigenfunctions of a commuting family of difference operators. We will have no need for these operators and so refer the reader to [Mac15, p. 322] for the full proof.

Theorem 3.3 (Macdonald Existence Theorem)**.** *For each partition λ there exists a unique symmetric function $P_\lambda(X;q,t) \in \Lambda_{\mathbb{F}}$ such that the following two conditions hold:*

1. *For some coefficients $a_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$ there holds*

$$P_\lambda(X;q,t) = m_\lambda(X) + \sum_{\mu<\lambda} a_{\lambda\mu}(q,t) m_\mu(X). \tag{3.1.9}$$

2. *One has*

$$\langle P_\lambda(X;q,t), P_\mu(X;q,t)\rangle = 0 \quad \textit{if} \quad \lambda \neq \mu. \tag{3.1.10}$$

Proof. We omit this proof. See [Mac15, p. 322]. ■

Several properties are immediate from the definition. If $\lambda = (1^r)$ for any $r \geqslant 0$ then

$$P_{(1^r)}(X;q,t) = m_{(1^r)}(X) = e_r(X). \tag{3.1.11}$$

In particular, the above definition tells us that the set $\{P_\lambda\}$ indexed over all partitions λ forms a basis for $\Lambda_{\mathbb{F}}$. Define the quantity $b_\lambda(q,t)$ by

$$b_\lambda = b_\lambda(q,t) = \frac{1}{\langle P_\lambda, P_\lambda\rangle}. \tag{3.1.12}$$

Then by letting

$$Q_\lambda(X;q,t) = b_\lambda(q,t) P_\lambda(X;q,t) \tag{3.1.13}$$

one has that

$$\langle P_\lambda, Q_\mu\rangle = \delta_{\lambda\mu}. \tag{3.1.14}$$

The identity

$$\sum_\lambda P_\lambda(X;q,t) Q_\lambda(X;q,t) = \prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty} \tag{3.1.15}$$

then follows immediately from Proposition 3.2. This is referred to as the Cauchy identity for Macdonald polynomials, and is foundational in the material to come. We also have the following important observation.

Proposition 3.4. *Let X be an alphabet of cardinality n. Then*

$$P_\lambda(X;q,t) = 0 \quad \textit{if } \ell(\lambda) > n. \tag{3.1.16}$$

Proof. By Proposition 2.2 if $\mu \leqslant \lambda$ then $\mu' \geqslant \lambda'$. As $\ell(\mu) = \mu'_1$ and $\ell(\lambda) = \lambda'_1$ there holds $\ell(\mu) \geqslant \ell(\lambda) > n$. Therefore by Definition 2.3, $m_\mu(x) = 0$ for all $\mu \leqslant \lambda$. ■

Specifying the parameters q and t leads to many other important classes of symmetric functions which further shows the fundamental importance of the Macdonald polynomials. Setting $q = t$ gives the Schur functions, written

$$P_\lambda(X; q, q) = s_\lambda(X).$$

The Schur functions have many differing descriptions, and play important roles in representation theory and algebraic combinatorics. There are two one-parameter families of symmetric functions obtainable from the Macdonald polynomials. There are the Hall–Littlewood polynomials, by setting $q = 0$, which again play a role in many areas of mathematics. For this thesis we will make continued use of the Jack symmetric functions, which are slightly more involved.

Definition 3.5. Let $t = q^\gamma$ for a positive real number γ. The *Jack symmetric function* indexed by the partition λ with parameter $1/\gamma$ is defined by

$$P_\lambda^{(1/\gamma)}(X) = \lim_{q \to 1} P_\lambda(X; q, q^\gamma). \tag{3.1.17}$$

Remark. Most authors define the Jack symmetric functions by setting $q = t^\alpha$ so that the indexing parameter is α. We will only encounter Jack symmetric functions whose parameter is $1/\gamma$, and so we define them in this manner.

The Jack symmetric functions satisfy many familiar properties, however with a single parameter $1/\gamma$ rather than the two arbitrary parameters q, t. For a comprehensive account see [Mac15, §10] and [Sta89].

3.2 Specialisation Formulae

Powerful in the theory of Macdonald polynomials are the numerous specialisation formulae, which will prove useful in later chapters. Formally, a specialisation by a partition μ with $\ell(\mu) \leqslant n$ of a polynomial $f(x_1, \dots, x_n)$ is a homomorphism $\mathbb{F}[x_1, \dots, x_n] \to \mathbb{F}$ for which $x_i \mapsto q^{\mu_i} t^{n-i}$. Each specialisation may be extended to the field $\mathbb{F}(x_1, \dots, x_n)$ for elements where the denominator is nonvanishing. The specialisation by the empty partition is called the *principal specialisation*, and is equivalent to the evaluation

$$f(1, t, t^2, \dots, t^{n-1}).$$

The principal specialisation for a symmetric function is more easily stated in plethystic notation. Here we identify

$$\frac{1 - t^n}{1 - t} = 1 + t + t^2 + \cdots + t^{n-1},$$

and so

$$P_\lambda(1, t, \dots, t^{n-1}; q, t) = P_\lambda\left[\frac{1-t^n}{1-t}\right].$$

Also, for a partition λ with $\ell(\lambda) \leqslant n$ define

$$\langle\lambda\rangle_n = q^{\lambda_1}t^{n-1} + q^{\lambda_2}t^{n-2} + \cdots + q^{\lambda_{n-1}}t + q^{\lambda_n}t^0. \tag{3.2.1}$$

With this convention the specialisation by a partition λ may be more easily written. For example if P_λ is a Macdonald polynomial then its specialisation by μ for $\ell(\mu) \leqslant n$ is written

$$P_\lambda[\langle\mu\rangle_n] = P_\lambda(q^{\mu_1}t^{n-1}, \dots, q^{\mu_{n-1}}t, q^{\mu_n}; q, t) = P_\lambda(\langle\mu\rangle_n). \tag{3.2.2}$$

Hence the principal specialisation may also be written

$$P_\lambda[\langle 0\rangle_n] = P_\lambda\left[\frac{1-t^n}{1-t}\right].$$

We note that the principal specialisation of a Macdonald polynomial is never identically zero where the Macdonald polynomial is nonzero. This follows from the explicit formula given in Proposition 3.8.

Specialisation of a Macdonald polynomial by a partition satisfies a beautiful symmetry in the indexing partition and the specialising partition. This was originally proved by Koornwinder in an unpublished manuscript, and hence is sometimes referred to as the Koornwinder–Macdonald evaluation symmetry (or Koornwinder–Macdonald duality) [Mac15, p. 332].

Proposition 3.6 (Koornwinder–Macdonald evaluation symmetry)**.** *Let λ, μ be partitions of length at most n. Then*

$$P_\lambda[\langle 0\rangle_n]P_\mu[\langle\lambda\rangle_n] = P_\mu[\langle 0\rangle_n]P_\lambda[\langle\mu\rangle_n]. \tag{3.2.3}$$

Remark. Our proof of this proposition hinges on the e-Pieri rule for Macdonald polynomials. For r a nonnegative integer and ν, ω partitions of length at most n this is

$$\frac{P_\omega e_r}{P_\omega[\langle 0\rangle_n]} = \sum_\nu B_{\nu/\omega}\frac{P_\nu}{P_\nu[\langle 0\rangle_n]} \tag{3.2.4}$$

where the sum is over all $\nu \supset \omega$ such that ν/ω is a vertical r-strip. Let I be the set of indices such that $\nu_i - \omega_i = 1$, i.e., the subset of $\{1, 2, \dots, n\}$ for which the row i in the diagram contains a square in the vertical r-strip ν/ω. Then the quantity $B_{\nu/\omega}$ is given by

$$B_{\nu/\omega} = t^{n(\nu)-n(\omega)} \prod_{\substack{i\in I,\\ j\notin I\\ i<j}} \frac{1-q^{\omega_i-\omega_j}t^{j-i+1}}{1-q^{\omega_i-\omega_j}t^{j-i}} \prod_{\substack{i\notin I\\ j\in I\\ i<j}} \frac{1-q^{\omega_i-\omega_j}t^{j-i-1}}{1-q^{\omega_i-\omega_j}t^{j-i}}. \tag{3.2.5}$$

In [Mac15, p. 322] Macdonald proves the Pieri formula (3.2.4) simultaneously with the evaluation symmetry (3.2.3). The proof of the Pieri formula is not instructive for what is to come, and so we assume the result and refer the reader to the aforementioned pages of Macdonald's book for the proof. Assuming the Pieri rule it is not hard to deduce the evaluation symmetry, and as this is a crucial part of our approach to generalised Selberg integrals below we produce the proof of it here. We also note that as in [Mac15, p. 332, 6.5] there holds

$$e_r[\langle\lambda\rangle_n]P_\lambda[\langle\mu\rangle_n] = \sum_\nu B_{\nu/\mu}P_\nu[\langle\lambda\rangle_n]. \tag{3.2.6}$$

This is a consequence of the action of the previously mentioned Macdonald operators on the polynomials P_λ.

Proof of Proposition 3.6. Let λ, μ be as in the statement of the proposition. We proceed by induction on $|\mu|$. When $\mu = 0$ both Macdonald polynomials indexed by μ equal 1 and the statement holds for any partition λ of length at most n.

Now assume that (3.2.3) is true for all λ with length at most n and all σ such that $|\sigma| < |\mu|$ or $|\sigma| = |\mu|$ and $\sigma < \mu$. Let ω be another partition with $|\omega| < |\mu|$. Making a plethystic substitution by λ in each symmetric function present in the Pieri rule (3.2.4) gives the equality

$$e_r[\langle\lambda\rangle_n]\frac{P_\omega[\langle\lambda\rangle_n]}{P_\omega[\langle 0\rangle_n]} = \sum_\nu B_{\nu/\omega}\frac{P_\nu[\langle\lambda\rangle_n]}{P_\nu[\langle 0\rangle_n]}.$$

By the inductive hypothesis we may interchange the partitions ω and λ. We may also apply the inductive hypothesis to interchange partitions wherever $\nu \neq \omega$. This yields the expansion

$$e_r[\langle\lambda\rangle_n]P_\lambda[\langle\omega\rangle_n] = B_{\mu/\omega}\frac{P_\mu[\langle\lambda\rangle_n]}{P_\mu[\langle 0\rangle_n]} + \sum_{\nu<\mu} B_{\nu/\omega}\frac{P_\nu[\langle\lambda\rangle_n]}{P_\nu[\langle 0\rangle_n]}. \tag{3.2.7}$$

Here the restriction that $\nu \supset \mu$ with ν/μ a vertical r-strip on the summation $\nu < \mu$ still holds. If we instead apply equation (3.2.6) we obtain

$$e_r[\langle\lambda\rangle_n]P_\lambda[\langle\omega\rangle_n] = B_{\mu/\omega}\frac{P_\lambda[\langle\mu\rangle_n]}{P_\lambda[\langle 0\rangle_n]} + \sum_{\nu<\mu} B_{\nu/\omega}\frac{P_\nu[\langle\lambda\rangle_n]}{P_\nu[\langle 0\rangle_n]}. \tag{3.2.8}$$

The only difference between (3.2.7) and (3.2.8) is that the roles of λ and μ in the leading terms have been swapped. Equating these expressions and cancelling like terms leaves the equation

$$\frac{P_\lambda[\langle\mu\rangle_n]}{P_\lambda[\langle 0\rangle_n]} = \frac{P_\mu[\langle\lambda\rangle_n]}{P_\mu[\langle 0\rangle_n]}. \qquad \blacksquare$$

This symmetry is a special case of a more general evaluation symmetry due to Warnaar.

Corollary 3.7 ([War10, Prop. 2.1])**.** *Let λ and μ be partitions satisfying $\ell(\lambda) \leqslant n$ and $\ell(\mu) \leqslant m$. Then for a parameter a there holds*

$$P_\mu\left[\frac{1-a}{1-t}\right] P_\lambda\left[at^{-m}\langle\mu\rangle_m + \frac{1-at^{-m}}{1-t}\right] \\ = P_\lambda\left[\frac{1-a}{1-t}\right] P_\mu\left[at^{-n}\langle\lambda\rangle_n + \frac{1-at^{-n}}{1-t}\right]. \tag{3.2.9}$$

Proof. We first prove the symmetric case with $n = m$. Make the scaling $a \mapsto at^n$ to obtain

$$P_\mu\left[\frac{1-at^n}{1-t}\right] P_\lambda\left[a\langle\mu\rangle_n + \frac{1-a}{1-t}\right] = P_\lambda\left[\frac{1-at^n}{1-t}\right] P_\mu\left[a\langle\lambda\rangle_n + \frac{1-a}{1-t}\right].$$

We may view (3.2.9) as an identity involving polynomials in the parameter a with coefficients in the field $\mathbb{F}$. Each side is also homogeneous of degree $|\lambda| + |\mu|$. Hence it suffices to prove the identity for $a = t^p$ for p a nonnegative integer. Assume this is the case, then for any symmetric function f we have

$$f\left[t^p\langle\lambda\rangle_n + \frac{1-t^p}{1-t}\right] = f[\langle\lambda\rangle_{n+p}].$$

From this we see that with $a = t^p$ (3.2.9) is equivalent to

$$P_\lambda\left[\frac{1-t^{n+p}}{1-t}\right] P_\mu[\langle\lambda\rangle_{n+p}] = P_\mu\left[\frac{1-t^{n+p}}{1-t}\right] P_\lambda[\langle\mu\rangle_{n+p}].$$

This is simply Proposition 3.6 for $(n+p)$-letter alphabets. Therefore (3.2.9) is a polynomial identity which is true for infinitely many values of a, and so the identity holds for general a. Now fix some integer m such that $\ell(\mu) \leqslant m \leqslant n$. Then

$$a\langle\mu\rangle_n = at^{n-m}\langle\mu\rangle_m + a\frac{1-t^{n-m}}{1-t}.$$

Therefore we may now write (3.2.9) as

$$P_\mu\left[\frac{1-at^n}{1-t}\right] P_\lambda\left[at^{n-m}\langle\mu\rangle_m + \frac{1-at^{n-m}}{1-t}\right] = P_\lambda\left[\frac{1-at^n}{1-t}\right] P_\mu\left[a\langle\lambda\rangle_n + \frac{1-a}{1-t}\right].$$

Now scale a by $a \mapsto at^{-n}$ to obtain

$$P_\mu\left[\frac{1-a}{1-t}\right] P_\lambda\left[at^{-m}\langle\mu\rangle_m + \frac{1-at^{-m}}{1-t}\right] = P_\lambda\left[\frac{1-a}{1-t}\right] P_\mu\left[at^{-n}\langle\lambda\rangle_n + \frac{1-at^{-n}}{1-t}\right].$$

This is symmetric in m and n, and so the restriction $m \leqslant n$ may be removed. ∎

The specialised Macdonald polynomials appearing in the evaluation symmetries stated above may in fact be evaluated explicitly. In order to state specialisation formulae succinctly we will need some standard notation from the theory of Macdonald polynomials. Recall the infinite q-shifted factorial (2.3.10). Then for $z \in \mathbb{C}$ we define the finite analogue by

$$(a;q)_z = \frac{(a;q)_\infty}{(aq^z;q)_\infty}. \tag{3.2.10}$$

Observe that if n is a nonnegative integer then

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

While these are often viewed as formal power series, we will sometimes require that q is a complex number such that $|q| < 1$ to resolve issues of convergence. The definition (3.2.10) may be extended to partitions, which gives a q,t-analogue of the q-shifted factorial. In order to do so, recall the arm length, leg length, and related definitions (2.1.3). Let λ be a partition and z an indeterminate. Define the q,t-shifted factorial indexed by λ to be

$$(z;q,t)_\lambda := \prod_{s\in\lambda}(1-zq^{a'(s)}t^{-l'(s)}) = \prod_{i=1}^{n}(zt^{1-i};q)_{\lambda_i}. \tag{3.2.11}$$

Here we may choose any $n \geqslant \ell(\lambda)$ as $\lambda_j = 0$ for $j > \ell(\lambda)$. We also have need of the generalised hook polynomials. Define these by

$$\begin{aligned} c_\lambda(q,t) &:= \prod_{s\in\lambda}(1-q^{a(s)}t^{l(s)+1}) && \text{(3.2.12a)}\\ &= \prod_{i=1}^{n}(t^{n-i+1};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\frac{(t^{j-i};q)_{\lambda_i-\lambda_j}}{(t^{j-i+1};q)_{\lambda_i-\lambda_j}},\\ c'_\lambda(q,t) &:= \prod_{s\in\lambda}(1-q^{a(s)+1}t^{l(s)}) && \text{(3.2.12b)}\\ &= \prod_{i=1}^{n}(qt^{n-i};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\frac{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}{(qt^{j-i};q)_{\lambda_i-\lambda_j}}. \end{aligned}$$

Again the choice of n is irrelevant as long as $n \geqslant \ell(\lambda)$. We can now state the formula for a principally specialised Macdonald polynomial.

Proposition 3.8. *For λ a partition of length at most n one has*

$$P_\lambda\Big(\Big[\frac{1-t^n}{1-t}\Big];q,t\Big) = P_\lambda\Big[\frac{1-t^n}{1-t}\Big] = t^{n(\lambda)}\frac{(t^n;q,t)_\lambda}{c_\lambda(q,t)}. \tag{3.2.13}$$

Proof. Restate the e-Pieri rule as (3.2.4)

$$P_\mu e_r = \sum_\lambda \psi'_{\lambda/\mu}P_\lambda \tag{3.2.14}$$

where the sum is again over partitions $\lambda \supset \mu$ with length at most n for which λ/μ is a vertical r-strip. The coefficient is given by

$$\psi'_{\lambda/\mu} = B_{\lambda/\mu} \frac{P_\mu[\langle 0 \rangle]}{P_\lambda[\langle 0 \rangle]}.$$

Assume that λ has the form $\lambda_i = \mu_i + 1$ for $1 \leqslant i \leqslant r$. That is, λ is obtained by adding a column of length r to the partition μ. In this case $\psi'_{\lambda/\mu} = 1$. To see this observe that the coefficient of x^λ on the left hand side of (3.2.14) is one, which follows from Theorem 3.3 and Definition 2.5. Hence the coefficient of x^λ is also one, which is precisely $\psi'_{\lambda/\mu}$. This allows us to write

$$P_\lambda[\langle 0 \rangle] = B_{\lambda/\mu} P_\mu[\langle 0 \rangle] \tag{3.2.15}$$

in this case. We may now begin the proof proper. When $\lambda = 0$ both sides of (3.2.13) are one, and so the equality holds. By (3.2.15) it suffices to show that

$$t^{n(\mu)-n(\lambda)} B_{\lambda/\mu} = \frac{(t^n;q,t)_\lambda c_\mu(q,t)}{(t^n;q,t)_\mu c_\lambda(q,t)}. \tag{3.2.16}$$

Denote the right hand side of (3.2.16) by RHS(3.2.16). Applying the definitions (3.2.11) and (3.2.12a) we see that

$$\text{RHS}(3.2.16) = \prod_{1 \leqslant i < j \leqslant n} \frac{(t^{j-i};q)_{\lambda_i - \lambda_j} (t^{j-i+1};q)_{\mu_i - \mu_j}}{(t^{j-i+1};q)_{\lambda_i - \lambda_j} (t^{j-i};q)_{\mu_i - \mu_j}}.$$

Let I be the set of indices such that $\lambda_i - \mu_i = 1$. Then the product splits into two products such that

$$\begin{aligned} \text{RHS}(3.2.16) = \prod_{\substack{i \in I \\ j \notin I \\ i<j}} & \frac{(t^{j-i};q)_{\lambda_i - \lambda_j} (t^{j-i+1};q)_{\mu_i - \mu_j}}{(t^{j-i+1};q)_{\lambda_i - \lambda_j} (t^{j-i};q)_{\mu_i - \mu_j}} \\ & \times \prod_{\substack{i \notin I \\ j \in I \\ i<j}} \frac{(t^{j-i};q)_{\lambda_i - \lambda_j} (t^{j-i+1};q)_{\mu_i - \mu_j}}{(t^{j-i+1};q)_{\lambda_i - \lambda_j} (t^{j-i};q)_{\mu_i - \mu_j}}. \end{aligned}$$

The cases where $i, j \in I$ and $i, j \notin I$ are both one, and so we ignore these. We know that $I = \{1, \ldots, r\}$, and so the second product in the above expression is empty. Therefore

$$\begin{aligned} \text{RHS}(3.2.16) &= \prod_{\substack{i \in I \\ j \notin I \\ i<j}} \frac{(t^{j-i};q)_{\mu_i - \mu_j + 1} (t^{j-i+1};q)_{\mu_i - \mu_j}}{(t^{j-i+1};q)_{\mu_i - \mu_j + 1} (t^{j-i};q)_{\mu_i - \mu_j}} \\ &= \prod_{\substack{i \in I \\ j \notin I \\ i<j}} \frac{1 - q^{\mu_i - \mu_j} t^{j-i+1}}{1 - q^{\mu_i - \mu_j} t^{j-i}}. \end{aligned}$$

Recall that from (3.2.5) we have

$$B_{\lambda/\mu} = t^{n(\lambda)-n(\mu)} \prod_{\substack{i\in I\\ j\notin I\\ i<j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i+1}}{1-q^{\mu_i-\mu_j}t^{j-i}} \prod_{\substack{i\notin I\\ j\in I\\ i<j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}}.$$

It is immediate that

$$\mathrm{LHS}(3.2.16) = \prod_{\substack{i\in I\\ j\notin I\\ i<j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i+1}}{1-q^{\mu_i-\mu_j}t^{j-i}} \prod_{\substack{i\notin I\\ j\in I\\ i<j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}}.$$

Again the second product here is empty, and so we have

$$\mathrm{LHS}(3.2.16) = \prod_{\substack{i\in I\\ j\notin I\\ i<j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i+1}}{1-q^{\mu_i-\mu_j}t^{j-i}}.$$

This completes the proof. ■

The formula for the principal specialisation of the Macdonald polynomials may be generalised to any plethystic substitution of the form $X \mapsto (1-u)/(1-t)$. Let us formalise this in the following proposition.

Proposition 3.9. *Let λ be a partition and a an indeterminate. Then*

$$P_\lambda\left[\frac{1-a}{1-t}\right] = t^{n(\lambda)} \frac{(a;q,t)_\lambda}{c_\lambda(q,t)}. \tag{3.2.17}$$

Proof. Both sides are polynomials in a with coefficients in the field $\mathbb{F}$. For any nonnegative integer $n > \ell(\lambda)$ we have that (3.2.17) holds for $a = t^n$ by Proposition 3.8. Therefore (3.2.17) is a polynomial identity that is true for infinitely many values of a, and so is true in general. ■

At several points later on we will need specialisation formulae for the Q-Macdonald polynomials. As these are a simple scalar multiple of the P_λ, it suffices to compute the coefficient $b_\lambda(q,t)$ (from (3.1.12)) in terms of generalised hook polynomials. In order to do so recall the involution ω from (2.2.10). We wish to introduce a q,t-analogue of this involution. Let u and v be two parameters. Then $\omega_{u,v}$ is defined by its action on the p_r as

$$\omega_{u,v}(p_r) = (-1)^{r-1}\frac{1-u^r}{1-v^r}p_r.$$

So for p_λ the action is

$$\omega_{u,v}(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda \prod_{i\geqslant 1} \frac{1-u^{\lambda_i}}{1-v^{\lambda_i}}.$$

It is clear that $\omega_{u,u}$ is simply ω, and $\omega_{v,u} = \omega_{u,v}^{-1}$. When $u = q$ and $v = t$ we have that

$$\omega_{q,t} P_\lambda(x; q, t) = Q_{\lambda'}(x; t, q). \tag{3.2.18}$$

Note that q and t are interchanged on the right hand side of this identity. For a proof of this statement, and other properties of $\omega_{u,v}$ see [Mac15, p. 312]. As with ω, we may interpret $\omega_{u,v}$ plethystically. Here the substitution is

$$X \mapsto -\epsilon X \frac{1-u}{1-v}.$$

We now compute $b_\lambda(q,t)$. Begin with the identity (3.2.18). Making a plethystic substitution by $X \mapsto (1-u)/(1-t)$ on both sides gives

$$\begin{aligned} P_{\lambda'}\left(\left[\frac{1-u}{1-q}\right]; t, q\right) &= \omega_{q,t} Q_\lambda\left(\left[\frac{1-u}{1-q}\right]; q, t\right) \\ &= (-1)^{|\lambda|} Q_\lambda\left(\left[\frac{1-u}{t-1}\right]; q, t\right). \end{aligned}$$

Pulling out a factor of $-t^{-1}$ and noting $Q_\lambda(X; q, t) = Q_\lambda(X; q^{-1}, t^{-1})$ we then have that

$$b_\lambda(q,t) = \frac{(-t)^{|\lambda|} P_{\lambda'}([\frac{1-u}{1-q}]; t, q)}{Q_\lambda([\frac{1-u}{1-t^{-1}}]; q^{-1}, t^{-1})}.$$

By the specialisation formula (3.2.17) we have

$$b_\lambda(q,t) = (-t)^{|\lambda|} q^{n(\lambda')} t^{n(\lambda)} \frac{(u; t, q)_{\lambda'} c_\lambda(q^{-1}, t^{-1})}{(u; q^{-1}, t^{-1})_\lambda c_{\lambda'}(t, q)}.$$

Alternatively using (3.2.11) and (3.2.12) along with (2.1.5) this may be expressed in terms of arms and legs as

$$b_\lambda(q,t) = (-t)^{|\lambda|} \prod_{s \in \lambda} t^{l'(s)} \frac{1 - q^{-a(s)} t^{-l(s)-1}}{1 - u q^{-a'(s)} t^{l'(s)}} \prod_{s \in \lambda'} q^{l'(s)} \frac{1 - u q^{-l'(s)} t^{a'(s)}}{1 - q^{l(s)+1} t^{a(s)}}.$$

The product over $s \in \lambda'$ may be turned into a product over $s \in \lambda$ by interchanging arm lengths and colengths with leg lengths and colengths. This yields

$$b_\lambda(q,t) = (-t)^{|\lambda|} \prod_{s \in \lambda} q^{a'(s)} t^{l'(s)} \frac{1 - q^{-a(s)} t^{-l(s)-1}}{1 - q^{a(s)+1} t^{l(s)}}.$$

Observe that

$$\sum_{s \in \lambda} a(s) = \sum_{s \in \lambda} a'(s), \quad \text{and} \quad \sum_{s \in \lambda} l(s) = \sum_{s \in \lambda} l'(s),$$

which are direct consequences of the definitions (2.1.3) and (2.1.4). Hence we have that

$$b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} = \frac{c_\lambda(q,t)}{c'_\lambda(q,t)}. \tag{3.2.19}$$

It then follows from Proposition 3.9 that

$$Q_\lambda\left(\left[\frac{1-u}{1-t}\right];q,t\right) = \frac{t^{n(\lambda)}(u;q,t)_\lambda}{c'_\lambda(q,t)}. \tag{3.2.20}$$

3.3 Skew Functions

Both the Schur functions and Hall–Littlewood polynomials have skew variants, indexed by skew shapes. In an analogous way one may define skew Macdonald polynomials. Here we will cover the basics needed to prove several important identities for skew Macdonald polynomials.

As the Macdonald polynomials form a basis for $\Lambda_{\mathbb{F}}$ the product of Macdonald polynomials may be expanded in this basis. The coefficients in this expansion are referred to as the q,t*-Littlewood–Richardson coefficients*, as they reduce to the classical Littlewood–Richardson coefficients, which play the same role in the Schur case ($q=t$). Formally the q,t-Littlewood–Richardson coefficient $f^\lambda_{\mu\nu} = f^\lambda_{\mu\nu}(q,t)$ is defined by

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda. \tag{3.3.1}$$

Observe that the left hand side is homogeneous of degree $|\mu|+|\nu|$, and therefore $f^\lambda_{\mu\nu}$ vanishes unless $|\lambda| = |\mu| + |\nu|$. Furthermore $f^\lambda_{\mu\nu} = 0$ unless $\mu \subset \lambda$ and $\nu \subset \lambda$ [Mac15, p. 344].

Definition 3.10. Let λ and μ be partitions. Define the skew Macdonald polynomials $Q_{\lambda/\mu}$ and $P_{\lambda/\mu}$ by

$$Q_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} Q_\nu \tag{3.3.2}$$

and

$$P_{\lambda/\mu} = \frac{b_\mu(q,t)}{b_\lambda(q,t)} Q_{\lambda/\mu}. \tag{3.3.3}$$

It is immediate from this definition that $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ both vanish unless $\mu \subset \lambda$. Furthermore both are homogeneous polynomials of degree $|\lambda| - |\mu|$.

The skew Macdonald polynomials give a nice formula for the Macdonald polynomial on the sum of two alphabets. To see this, note that with the use of the Cauchy identity (3.1.15) and the above definitions we have

$$\begin{aligned}\sum_{\lambda} Q_{\lambda/\mu}(X)P_\lambda(Y) &= \sum_{\lambda,\nu} f^{\lambda}_{\mu\nu} Q_\nu(X)P_\lambda(Y) \\ &= \sum_{\nu} Q_\nu(X)P_\mu(Y)P_\nu(Y) \\ &= P_\mu(Y)\sigma\Big[XY\frac{1-t}{1-q}\Big].\end{aligned} \tag{3.3.4}$$

Using this we may now evaluate the double sum for alphabets X, Y, Z

$$\begin{aligned}\sum_{\lambda,\mu} Q_{\lambda/\mu}(X)P_\lambda(Y)Q_\mu(Z) &= \sum_{\mu} P_\mu(Y)Q_\mu(Z)\sigma\Big[XY\frac{1-t}{1-q}\Big] \\ &= \sigma\Big[YZ\frac{1-t}{1-q}\Big]\sigma\Big[XY\frac{1-t}{1-q}\Big] \\ &= \sigma\Big[Y(X+Z)\frac{1-t}{1-q}\Big] \\ &= \sum_{\lambda} P_\lambda(Y)Q_\lambda[X+Y].\end{aligned}$$

Extracting the coefficient of $P_\lambda(Y)$ on both sides gives

$$Q_\lambda[X+Z] = \sum_{\mu} Q_{\lambda/\mu}(X)Q_\mu(Z), \tag{3.3.5}$$

and by (3.3.3),

$$P_\lambda[X+Z] = \sum_{\mu} P_{\lambda/\mu}(X)P_\mu(Z) \tag{3.3.6}$$

Both of the above formulae are symmetric in the alphabets X, Z. We may now prove the following summation formula for skew functions.

Theorem 3.11. *Let $k \in \mathbb{N}$ and $\ell \in \mathbb{N} \cup \{\infty\}$. For partitions λ, μ such that $\ell(\lambda) \leqslant k$ and $\ell(\mu) \leqslant \ell$ we have that*

$$\begin{aligned}&\sum_{\nu} t^{-|\nu|} P_{\mu/\nu}\Big[\frac{1-a}{1-t}\Big] Q_{\lambda/\nu}\Big[\frac{1-q/at}{1-t}\Big] \\ &\qquad = t^{-k|\mu|} P_\mu\Big[\frac{1-at^k}{1-t}\Big] Q_\lambda\Big[\frac{1-qt^{\ell-1}/a}{1-t}\Big] \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}}.\end{aligned} \tag{3.3.7}$$

Here we adopt the convention that for infinite ℓ, $t^\ell := 0$.

Remark. For $k = \ell \in \mathbb{N}$ this is the identity

$$\begin{aligned}&\sum_{\nu} t^{-|\nu|} P_{\mu/\nu}\Big[\frac{1-a}{1-t}\Big] Q_{\lambda/\nu}\Big[\frac{1-q/at}{1-t}\Big] \\ &\qquad = t^{-k|\mu|} P_\mu\Big[\frac{1-at^k}{1-t}\Big] Q_\lambda\Big[\frac{1-qt^{k-1}/a}{1-t}\Big] \prod_{i,j=1}^{k} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}},\end{aligned} \tag{3.3.8}$$

which is [War10, Theorem 3.4]. For infinite ℓ the identity becomes

$$\sum_{\nu} t^{-|\nu|} P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] Q_{\lambda/\nu}\left[\frac{1-q/at}{1-t}\right] \tag{3.3.9}$$
$$= t^{-k|\mu|} P_{\mu}\left[\frac{1-at^k}{1-t}\right] Q_{\lambda}\left[\frac{1}{1-t}\right] \prod_{i=1}^{k} \prod_{j\geqslant 1} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}},$$

noting that

$$Q_{\lambda}\left[\frac{1}{1-t}\right] = \frac{t^{n(\lambda)}}{c'_{\lambda}(q,t)}$$

by (3.2.20).

Proof of Theorem 3.11. The proof proceeds in two parts. We first establish the result for $k = \ell \in \mathbb{N}$. Recall that for partitions λ and μ of length at most k the generalised evaluation symmetry of Corollary 3.7. We may write this in the case $k = n = m$ and scaling by $a \mapsto t^k$ as

$$P_{\lambda}\left[\frac{1-at^k}{1-t}\right] P_{\mu}\left[a\langle\lambda\rangle_k + \frac{1-a}{1-t}\right] = P_{\mu}\left[\frac{1-at^k}{1-t}\right] P_{\lambda}\left[a\langle\mu\rangle_k + \frac{1-a}{1-t}\right].$$

Using (3.3.6) the left-hand side may alternatively be expressed as

$$\sum_{\nu} P_{\lambda}\left[\frac{1-at^k}{1-t}\right] P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] P_{\nu}[a\langle\lambda\rangle_k] = P_{\mu}\left[\frac{1-at^k}{1-t}\right] P_{\lambda}\left[a\langle\mu\rangle_k + \frac{1-a}{1-t}\right].$$

For an arbitrary alphabet X we multiply both sides by $Q_{\lambda}(X)$ and sum over λ to obtain

$$\sum_{\lambda,\nu} P_{\lambda}\left[\frac{1-at^k}{1-t}\right] Q_{\lambda}(X) P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] P_{\nu}(a\langle\lambda\rangle_k)$$
$$= \sum_{\lambda} P_{\mu}\left[\frac{1-at^k}{1-t}\right] Q_{\lambda}(X) P_{\lambda}\left[a\langle\mu\rangle_k + \frac{1-a}{1-t}\right].$$

An application of the Cauchy identity (3.1.15) to the right hand side gives

$$\sum_{\lambda,\nu} P_{\lambda}\left[\frac{1-at^k}{1-t}\right] Q_{\lambda}(X) P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] P_{\nu}(a\langle\lambda\rangle_k)$$
$$= P_{\mu}\left[\frac{1-at^k}{1-t}\right] \sigma\left[a\langle\mu\rangle_k X \frac{1-t}{1-q}\right] \sigma\left[X\frac{1-a}{1-q}\right].$$

We now relabel the index λ by ω and make the substitution $X \mapsto b\langle\lambda\rangle_k$ to yield

$$\sum_{\nu,\omega} Q_{\omega}\left[\frac{1-at^k}{1-t}\right] P_{\omega}(b\langle\lambda\rangle_k) P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] P_{\nu}(a\langle\omega\rangle_k) \tag{3.3.10}$$
$$= P_{\mu}\left[\frac{1-at^k}{1-t}\right] \sigma\left[ab\langle\lambda\rangle_k\langle\mu\rangle_k \frac{1-t}{1-q}\right] \sigma\left[b\langle\lambda\rangle_k \frac{1-a}{1-q}\right].$$

Denote the left hand side of this by LHS(3.3.10). Using the homogeneity of the Macdonald polynomials together with the classical evaluation symmetry (Theorem (3.6)) we see that

$$\begin{aligned}\mathrm{LHS}(3.3.10) = \sum_{\nu,\omega} a^{|\nu|} b^{|\omega|} Q_\omega \left[\frac{1-at^k}{1-t}\right] P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] \\ \times \frac{P_\omega(\langle 0\rangle_k)}{P_\lambda(\langle 0\rangle_k)} P_\lambda(\langle\omega\rangle_k) P_\nu(\langle\omega\rangle_{k)}).\end{aligned}$$

Using the q,t-Littlewood–Richardson coefficients we may write

$$P_\lambda(\langle\omega\rangle_k)P_\nu(\langle\omega\rangle_k) = \sum_\eta f^\eta_{\lambda\nu} P_\eta(\langle\omega\rangle_k),$$

and so

$$\mathrm{LHS}(3.3.10) = \sum_{\eta,\nu,\omega} a^{|\nu|} b^{|\omega|} f^\eta_{\lambda\nu} Q_\omega\left[\frac{1-at^k}{1-t}\right] P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] \frac{P_\omega(\langle 0\rangle_k)}{P_\lambda(\langle 0\rangle_k)} P_\eta(\langle\omega\rangle_k).$$

Another application of the evaluation symmetry yields

$$\mathrm{LHS}(3.3.10) = \sum_{\eta,\nu,\omega} a^{|\nu|} b^{|\omega|} f^\eta_{\lambda\nu} Q_\omega\left[\frac{1-at^k}{1-t}\right] P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] \frac{P_\eta(\langle 0\rangle_k)}{P_\lambda(\langle 0\rangle_k)} P_\omega(\langle\eta\rangle_k).$$

The sum over ω may be evaluated by the Cauchy identity (3.1.15) and hence

$$\mathrm{LHS}(3.3.10) = \sum_{\eta,\nu} a^{|\nu|} f^\eta_{\lambda\nu} P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] \frac{P_\eta(\langle 0\rangle_k)}{P_\lambda(\langle 0\rangle_k)} \sigma\left[b\langle\eta\rangle_k \frac{1-at^k}{1-q}\right].$$

Equating this with the right hand side of (3.3.10) we now have the identity

$$\begin{aligned}&\sum_{\eta,\nu} a^{|\nu|} f^\eta_{\lambda\nu} P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] \frac{P_\eta(\langle 0\rangle_k)}{P_\lambda(\langle 0\rangle_k)} \sigma\left[b\langle\eta\rangle_k \frac{1-at^k}{1-q}\right] \\ &\qquad = P_\mu\left[\frac{1-at^k}{1-t}\right] \sigma\left[ab\langle\lambda\rangle_k\langle\mu\rangle_k \frac{1-t}{1-q}\right] \sigma\left[b\langle\lambda\rangle_k \frac{1-a}{1-q}\right].\end{aligned}$$

In terms of q-shifted factorials this is

$$\begin{aligned}&\sum_{\eta,\nu} a^{|\nu|} f^\eta_{\lambda\nu} P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] \frac{P_\eta(\langle 0\rangle_k)}{P_\lambda(\langle 0\rangle_k)} \prod_{i=1}^k \frac{(abq^{\eta_i}t^{2k-i};q)_\infty}{(bq^{\eta_i}t^{k-i};q)_\infty} \\ &\qquad = P_\mu\left[\frac{1-at^k}{1-t}\right] \prod_{i,j=1}^k \frac{(abq^{\lambda_i+\mu_j}t^{2k-i-j+1};q)_\infty}{(abq^{\lambda_i+\mu_j}t^{2k-i-j};q)_\infty} \prod_{i=1}^k \frac{(abq^{\lambda_i}t^{k-i};q)_\infty}{(bq^{\lambda_i}t^{k-i};q)_\infty}.\end{aligned}$$

Making some elementary manipulations of q-shifted factorials and scaling b by $b \mapsto bt^{1-k}$ we see that

$$\sum_{\eta,\nu} a^{|\nu|} \frac{(b;q,t)_\eta}{(abt^k;q,t)_\eta} f^{\eta}_{\lambda\nu} P_{\mu/\nu}\left[\frac{1-a}{1-t}\right] P_\eta(\langle 0\rangle_k) \tag{3.3.11}$$

$$= \frac{(b;q,t)_\lambda}{(ab;q,t)_\lambda} P_\lambda(\langle 0\rangle_k) P_\mu\left[\frac{1-at^k}{1-t}\right] \prod_{i,j=1}^{k} \frac{(abt^{k-i-j+1};q)_{\lambda_i+\mu_j}}{(abt^{k-i-j+2};q)_{\lambda_i+\mu_j}}.$$

Now set $b = q^{-N}$. We wish to replace the partitions λ and η by $\hat{\lambda}$ and $\hat{\eta}$ such that $\hat{\lambda}_i = N - \lambda_{k-i+1}$, and similarly for $\hat{\eta}$. These are the complements in the rectangular partition (N^k) of λ and η. Hence we must take N to be such that $\max\{\lambda_1, \eta_1\} \leqslant N$. In order to trade the new complemented partitions for the original λ and η we need several formulae. Note that the q,t-shifted factorial indexed by $\hat{\lambda}$ may be written as

$$(a;q,t)_{\hat{\lambda}} = (-q/a)^{|\lambda|} t^{n(\lambda)} q^{n(\lambda')-N|\lambda|} \frac{(a;q,t)_{(N^k)}}{(q^{1-N}t^{k-1}/a;q)_\lambda}.$$

Therefore the ratios occurring in our equation take the form

$$\frac{(a;q,t)_{\hat{\lambda}}}{(b;q,t)_{\hat{\lambda}}} = (b/a)^{|\lambda|} \frac{(a;q,t)_{(N^k)}}{(b;q,t)_{(N^k)}} \frac{(q^{1-N}t^{k-1}/b;q)_\lambda}{(q^{1-N}t^{k-1}/a;q)_\lambda}.$$

One may also write the principally specialised Macdonald polynomial $P_{\hat{\lambda}}(\langle 0\rangle_k)$ in terms of $P_\lambda(\langle 0\rangle_k)$ as

$$P_{\hat{\lambda}}(\langle 0\rangle_k) = t^{N\binom{k}{2}-(k-1)|\lambda|} P_\lambda(\langle 0\rangle_k).$$

We also note that we may write the $f^{\hat{\eta}}_{\hat{\lambda}\nu}(q,t)$ in terms of $f^{\lambda}_{\eta\nu}$ as

$$f^{\hat{\eta}}_{\hat{\lambda}\nu}(q,t) = \frac{Q_\eta[\frac{1-qt^{k-1}}{1-t}]}{Q_\lambda[\frac{1-qt^{k-1}}{1-t}]} \frac{P_\lambda(\langle 0\rangle_k)}{P_\eta(\langle 0\rangle_k))} f^{\lambda}_{\eta\nu}.$$

These last two results are given in [War05, p. 259, 263]. The above identities along with a wealth of others may be deduced from [Rai05, §2]. Returning to (3.3.11) after making the above replacements and applying the identities given we now have the identity

$$\sum_{\eta,\nu} t^{-|\nu|} f^{\lambda}_{\eta\nu} Q_\eta\left[\frac{1-q/at}{1-t}\right] P_{\mu/\nu}\left[\frac{1-a}{1-t}\right]$$

$$= t^{-k|\mu|} Q_\lambda\left[\frac{1-qt^{k-1}/a}{1-t}\right] P_\mu\left[\frac{1-at^k}{1-t}\right] \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}}.$$

Here we have also used the fact that the summand vanishes unless $|\nu| + |\eta| = |\lambda|$. Now the sum over η may be performed by the definition (3.3.2) to yield

$$\sum_{\nu} t^{-|\nu|} Q_{\lambda/\nu}\Big[\frac{1-q/at}{1-t}\Big] P_{\mu/\nu}\Big[\frac{1-a}{1-t}\Big] = t^{-k|\mu|} Q_{\lambda}\Big[\frac{1-qt^{k-1}/a}{1-t}\Big] P_{\mu}\Big[\frac{1-at^k}{1-t}\Big] \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}}.$$

This is precisely (3.3.8), and so the proof is complete for $k = \ell$.

With the case $k = \ell$ established it is not hard to produce the more general result. Observe that for $k = \ell$ we have that

$$\begin{aligned}\prod_{i,j=1}^{k} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}} &= \prod_{i=1}^{k}\prod_{j=1}^{\ell(\mu)} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}} \prod_{i=1}^{k} \frac{(qt^{\ell(\mu)-i}/a;q)_{\lambda_i}}{(qt^{k-i}/a;q)_{\lambda_i}} \\ &= \frac{(qt^{\ell(\mu)-1};q,t)_{\lambda}}{(qt^{k-1};q,t)_{\lambda}} \prod_{i=1}^{k}\prod_{j=1}^{\ell(\mu)} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}} \\ &= \frac{Q_{\lambda}[\frac{1-qt^{\ell(\mu)-1}/a}{1-t}]}{Q_{\lambda}[\frac{1-qt^{k-1}/a}{1-t}]} \prod_{i=1}^{k}\prod_{j=1}^{\ell(\mu)} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}}.\end{aligned}$$

This implies that (3.3.8) is equivalent to

$$\sum_{\nu} t^{-|\nu|} P_{\mu/\nu}\Big[\frac{1-a}{1-t}\Big] Q_{\lambda/\nu}\Big[\frac{1-q/at}{1-t}\Big] = t^{-k|\mu|} P_{\mu}\Big[\frac{1-at^k}{1-t}\Big] Q_{\lambda}\Big[\frac{1-qt^{\ell(\mu)-1}/a}{1-t}\Big] \prod_{i=1}^{k}\prod_{j=1}^{\ell(\mu)} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}}. \tag{3.3.12}$$

Now for ℓ any integer such that $\ell(\mu) \leqslant \ell$ we also have that

$$\begin{aligned}\prod_{i=1}^{k}\prod_{j\geqslant 1} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}} &= \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}} \prod_{i=1}^{k} (qt^{\ell-i}/a;q)_{\lambda_i} \\ &= (qt^{\ell-1}/a;q,t)_{\lambda} \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}} \\ &= \frac{Q_{\lambda}[\frac{1-qt^{\ell-1}/a}{1-t}]}{Q_{\lambda}[\frac{1}{1-t}]} \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}}.\end{aligned} \tag{3.3.13}$$

Taking $\ell = \ell(\mu)$ here shows that the case $k = \ell$ (3.3.8) and the case $\ell = \infty$ (3.3.9) are equivalent. However by the considerations (3.3.13) we may conclude that (3.3.7) holds for any $\ell \in \mathbb{N} \cup \{\infty\}$ with $\ell(\mu) \leqslant \ell$. ■

We require one last identity for skew functions, which is a corollary of the above theorem.

Corollary 3.12. *For partitions λ and μ such that $\ell(\lambda) \leqslant k \leqslant \ell$ there holds*

$$\sum_{\nu} t^{-|\nu|} P_{\mu/\nu}(\langle 0\rangle_{\ell-k}) Q_{\lambda/\nu}\left[\frac{1-qt^{k-\ell-1}}{1-t}\right] \tag{3.3.14}$$

$$= t^{-k|\mu|} P_{\mu}(\langle 0\rangle_{\ell}) Q_{\lambda}\left[\frac{1-qt^{k-1}}{1-t}\right] \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(qt^{k-\ell+j-i-1};q)_{\lambda_i-\mu_j}}{(qt^{k-\ell+j-i};q)_{\lambda_i-\mu_j}}.$$

Proof. Fix $k \leqslant \ell \in \mathbb{N}$ in Theorem 3.11. Then setting $a^{\ell-k}$ gives (3.3.14). Note that the restriction $\ell(\mu) \leqslant \ell$ inherited from the lemma is unnecessary as both sides vanish if this is not the case. Clearly the right-hand side of (3.3.14) vanishes for $\ell > \ell(\mu)$ due to the factor

$$P_{\mu}(\langle 0\rangle_{\ell}).$$

The left-hand side will vanish unless $\nu \subseteq \lambda$ due to the presence of

$$Q_{\lambda/\nu}\left[\frac{1-qt^{k-\ell-1}}{1-t}\right].$$

This implies that $\ell(\nu) \leqslant \ell(\lambda) \leqslant k$ (a restriction on the sum). Now the factor

$$P_{\mu/\nu}(\langle 0\rangle_{\ell-k})$$

vanishes unless $\ell(\mu) - \ell(\nu) \leqslant \ell - k$. This gives that $\ell(\mu) \leqslant \ell - k + \ell(\nu) \leqslant \ell$. Therefore both sides of (3.3.14) vanish for $\ell(\mu) > \ell$ and so we may remove the restriction that $\ell(\mu) \leqslant \ell$. ■

3.4 Cauchy Identities

We now aim to apply the results of the previous section in order to prove two new Cauchy-type identities for Macdonald polynomials. For notational convenience we will suppress q,t-dependence from Macdonald polynomials, shifted factorials, and generalised hook polynomials in this section. The Cauchy identity (3.1.15) states that for X and Y arbitrary alphabets there holds

$$\sum_{\lambda} P_{\lambda}(X) Q_{\lambda}(Y) = \prod_{x\in X}\prod_{y\in Y} \frac{(txy)_{\infty}}{(xy)_{\infty}},$$

where the sum is over all partitions. This has a natural association to $\mathfrak{sl}_2$ (or A_1, we use these interchangeably) basic hypergeometric series as it is a generalisation of the Kaneko–Macdonald q-binomial theorem:

$$\sum_{\lambda} \frac{t^{n(\lambda)}(a)_{\lambda}}{c'_{\lambda}} P_{\lambda}(X) = \prod_{x\in X} \frac{(ax)_{\infty}}{(x)_{\infty}}.$$

The latter obtained by specialising $Y \mapsto (1-a)/(1-t)$ in the Cauchy identity. For a full proof see Theorem 4.2.3 below. This may be generalised to $\mathfrak{sl}_{n+1}$. Indeed we have the following summation formula due to Warnaar.

Theorem 3.13 ([War09, Theorem 3.2]). *Let $X^{(1)}, \dots, X^{(n)}$ be n alphabets such that*

$$X^{(1)} = (x_1^{(1)}, \dots, x_{k_1}^{(1)}), \qquad X^{(s)} = z_s \frac{1-t^{k_s}}{1-t}, \qquad 2 \leqslant s \leqslant n$$

for integers $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$. Then

$$\begin{aligned} \sum_{\lambda^{(1)},\dots,\lambda^{(n)}} & \frac{(a)_{\lambda^{(n)}}}{(qt^{k_n-1})_{\lambda^{(n)}}} \prod_{s=1}^{n} \Big[t^{n(\lambda^{(s)})} \frac{(qt^{k_s-1})_{\lambda^{(s)}}}{c'_{\lambda^{(s)}}} P_{\lambda^{(s)}}(X^{(s)}) \Big] \qquad (3.4.1) \\ & \times \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \prod_{j=1}^{k_{s+1}} \frac{(qt^{j-i+k_s-k_{s+1}-1})_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}}{(qt^{j-i+k_s-k_{s+1}})_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}} \\ = \prod_{i=1}^{k_1} & \Big[\frac{(az_2\cdots z_n x_i^{(1)} t^{k_1+\cdots+k_{n-1}-n+1})_\infty}{(z_2\cdots z_n x_i^{(1)} t^{k_1+\cdots+k_{n-1}-n+1})_\infty} \\ & \quad \times \prod_{r=1}^{n-1} \frac{(qz_2\cdots z_r x_i^{(1)} t^{k_1+\cdots+k_r-k_{r+1}-r})_\infty}{(z_2\cdots z_r x_i^{(1)} t^{k_1+\cdots+k_r-k_{r+1}-r})_\infty} \Big] \\ & \times \prod_{s=2}^{n} \prod_{i=1}^{k_s-k_{s-1}} \frac{(az_s\cdots z_n t^{i+s+k_{s-1}+\cdots+k_{n-1}-n-1})_\infty}{(z_s\cdots z_n t^{i+s+k_{s-1}+\cdots+k_{n-1}-n-1})_\infty} \\ & \times \prod_{2\leqslant s\leqslant r\leqslant n-1} \prod_{i=1}^{k_s-k_{s-1}} \frac{(qz_s\cdots z_r t^{i+s-r+k_{s-1}+\cdots+k_r-k_{r+1}-2})_\infty}{(z_s\cdots z_r t^{i+s-r+k_{s-1}+\cdots+k_r-1})_\infty}. \end{aligned}$$

Here the sum is over partitions $\lambda^{(1)}, \dots, \lambda^{(n)}$ such that $\ell(\lambda^{(s)}) \leqslant k_s$ for $1 \leqslant s \leqslant n$ and

$$\lambda_i^{(s)} \geqslant \lambda_{i-k_s+k_{s+1}}^{(s+1)} \qquad \text{for } 1 \leqslant i \leqslant k_s. \tag{3.4.2}$$

Before continuing some comments on the conditions (3.4.2) are in order. Fix $0 \leqslant k \leqslant \ell$ nonnegative integers and let λ, μ be partitions such that $\ell(\lambda) \leqslant k$ and $\ell(\mu) \leqslant \ell$. The summand on the left-hand side of (3.4.1) contains products of the form

$$\prod_{i=1}^{k} \prod_{j=1}^{\ell} \frac{(qt^{j-i+k-\ell+1})_{\lambda_i-\mu_j}}{(qt^{j-i+k-\ell})_{\lambda_i-\mu_j}}.$$

When the power of t is zero this is in danger of being undefined as the difference $\lambda_i - \mu_j$ may be negative. In this case

$$(q)_{-n} = \frac{1}{(q^{1-n})_n} = \frac{1}{(1-q^{1-n})\cdots(1-q^{-1})(1-q^0)},$$

which is undefined. This is not a problem thanks to the following lemma.

Lemma 3.14. *Let λ and μ be partitions and k, ℓ nonnegative integers with $k \leqslant \ell$. Then*

$$f^{\ell,k}_{\lambda,\mu}(q,t) := \lim_{a\to 1} \prod_{i=1}^{k} \prod_{j=1}^{\ell} \frac{(aqt^{j-i+k-\ell-1})_{\lambda_i-\mu_j}}{(aqt^{j-i+k-\ell})_{\lambda_i-\mu_j}} \tag{3.4.3}$$

is well-defined. Furthermore a necessary and sufficient condition for nonvanishing is

$$\lambda_i \geqslant \mu_{i-k+\ell}, \qquad \text{for } 1 \leqslant i \leqslant k. \tag{3.4.4}$$

Remark. The condition (3.4.4) may be represented as

$$\begin{array}{ccccccccccccccc} & & & & \lambda_1 & \geqslant & \lambda_2 & \geqslant & \cdots & \geqslant & \lambda_k & \geqslant & \lambda_{k+1} & \geqslant & \cdots \geqslant 0 \\ & & & & \veebar & & \veebar & & & & \veebar & & & & \\ \mu_1 & \geqslant & \cdots & \geqslant & \mu_{\ell-k+1} & \geqslant & \mu_{\ell-k+2} & \geqslant & \cdots & \geqslant & \mu_\ell & \geqslant & \mu_{\ell+1} & \geqslant & \cdots \geqslant 0. \end{array}$$

Proof of Lemma 3.14. In order to check the limit is well-defined note that for fixed i the powers of t in (3.4.3) are zero when $j = i-k+\ell+1$ in the numerator and $j = i-k+\ell$ in the denominator. We also require that $j \leqslant \ell$ and so in the numerator $k-\ell \leqslant i \leqslant k-1$ and in the denominator $k-\ell+1 \leqslant i \leqslant k$. As $k \leqslant \ell$ both lower bounds are automatically satisfied. Therefore taking the product of the t-independent q-shifted factorials in (3.4.3) yields

$$\frac{\prod_{i=1}^{k-1}(aq)_{\lambda_i-\mu_{i-k+\ell+1}}}{\prod_{i=1}^{k}(aq)_{\lambda_i-\mu_{i-k+\ell}}} = \frac{1}{(aq)_{\lambda_k-\mu_\ell}} \prod_{i=1}^{k-1} (aq^{1+\lambda_i-\mu_{i-k+\ell}})_{\mu_{i-k+\ell}-\mu_{i-k+\ell+1}}.$$

Making a shift in the indices gives

$$\frac{\prod_{i=1}^{k-1}(aq)_{\lambda_i-\mu_{i-k+\ell+1}}}{\prod_{i=1}^{k}(aq)_{\lambda_i-\mu_{i-k+\ell}}} = \frac{1}{(aq)_{\lambda_k-\mu_\ell}} \prod_{i=\ell-k+1}^{\ell-1} (aq^{1+\lambda_{i+\ell-k}-\mu_i})_{\mu_i-\mu_{i+1}}. \tag{3.4.5}$$

Now $\mu_i \geqslant \mu_{i+1}$ as μ is a partition, and so limit as $a \to 1$ exists.

As the limit is well-defined we consider the result of taking said limit. We also consider only the t-independent terms when checking the vanishing conditions. Clearly the term $1/(q)_{\lambda_k-\mu_\ell}$ will vanish unless $\lambda_k \geqslant \mu_\ell$. Now for fixed i a term in the product on the right of (3.4.5) vanishes for $\lambda_{i-k+\ell} < \mu_i$ and $\mu_i > \mu_{i+1}$. We are interested in non-vanishing. Hence we note that in order for the product in (3.4.5) to be nonvanishing one of the following must hold for all $\ell-k+1 \leqslant i \leqslant \ell-1$:

$$\lambda_{i+k-\ell} \geqslant \mu_i, \tag{3.4.6}$$

$$\lambda_{i+k-\ell} < \mu_i = \mu_{i+1}. \tag{3.4.7}$$

Now assume that $\lambda_k \geqslant \mu_\ell$ and one of (3.4.6) and (3.4.7) holds. Consider the largest i for which (3.4.7) holds but (3.4.6) does not. We cannot have $i = \ell-1$ as this would imply

$$\lambda_k < \mu_{\ell-1} = \mu_\ell$$

contradicting $\lambda_k \geqslant \mu_\ell$. In a similar fashion no such maximal i exists with $i \leqslant \ell - 1$ as we then get

$$\lambda_{i+k-\ell} < \mu_i = \mu_{i+1}.$$

However as (3.4.6) must now hold for $i+1$ this would give $\lambda_{i+k-\ell} < \lambda_{i+k-\ell+1}$ which contradicts λ being a partition. Therefore we conclude that (3.4.6) must hold for all $\ell - k + 1 \leqslant i \leqslant \ell - 1$. This is equivalent to the desired conditions by a shift of indices, and so we are done. ■

In summary, Lemma 3.14 implies that the summand of (3.4.1) vanishes unless the conditions (3.4.2) hold. Therefore these conditions are unnecessary and may be removed.

In [War09] the above is phrased as an $\mathfrak{sl}_{n+1}$ q-binomial theorem for Macdonald polynomials. We interpret this as a specialised Cauchy-type identity. To see this, observe that in view of the principal specialisation formula (3.2.20) we may write

$$\begin{aligned}\frac{(a)_{\lambda^{(n)}}}{(qt^{k_n-1})_{\lambda^{(n)}}} &\prod_{s=1}^{n}\left[t^{n(\lambda^{(s)})}\frac{(qt^{k_s-1})_{\lambda^{(s)}}}{c'_{\lambda^{(s)}}}P_{\lambda^{(s)}}(X^{(s)})\right]\\ &= P_{\lambda^{(1)}}(X^{(s)})Q_{\lambda^{(n)}}\left[\frac{1-a}{1-t}\right]\prod_{s=1}^{n-1}P_{\lambda^{(s+1)}}\left[z_{s+1}\frac{1-t^{k_{s+1}}}{1-t}\right]Q_{\lambda^{(s)}}\left[\frac{1-qt^{k_s-1}}{1-t}\right].\end{aligned}$$

This is the product of $2n$ Macdonald polynomials, of which all but one are specialised. The presence of the polynomial $Q_{\lambda^{(n)}}[\frac{1-a}{1-t}]$ indicates that it should be possible to interpret (3.4.1) in the ring of symmetric functions on infinitely many variables in some alphabet Y. When making the plethystic substitution $Y \mapsto (1-a)/(1-t)$ (3.4.1) should be recovered. This is indeed possible, and in fact there are two such Cauchy-type formulas.

Theorem 3.15 (A$_n$ Cauchy-type formula I)**.** *Let $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_{n-1}$ be nonnegative integers and $k_n \in \mathbb{N}\cup\{\infty\}$. Furthermore let a_{n-1} be an indeterminate and $a_r := t^{k_r - k_{r+1}}$ for $1 \leqslant r \leqslant n-2$. Define the alphabets $X^{(1)}, \dots, X^{(n)}$ and $Y^{(1)}, \dots, Y^{(n)}$ by*

$$\begin{aligned} X^{(1)} &:= \{x_1, \dots, x_{k_1}\}, & X^{(r+1)} &:= \frac{t^{-k_r} - a_r^{-1}}{1-t}, & 1 \leqslant r \leqslant n-1\\ Y^{(n)} &:= \{y_1, \dots, y_{k_n}\}, & Y^{(r)} &:= z_r\frac{t - a_r q t^{k_{r+1}}}{1-t}, & 1 \leqslant r \leqslant n-1.\end{aligned}$$

Then

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \prod_{r=1}^{n} P_{\lambda^{(r)}}(X^{(r)}) Q_{\lambda^{(r)}}(Y^{(r)}) \prod_{r=1}^{n-1} \prod_{i=1}^{k_r} \prod_{j=1}^{k_{r+1}} \frac{(a_r q t^{j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(a_r q t^{j-i})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}} \tag{3.4.8}$$
$$= \prod_{s=1}^{n-1} \prod_{x\in X} \frac{(a_s q z_1\cdots z_s x)_\infty}{(t z_1 \cdots z_s x)_\infty} \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1}\cdots z_s)_\infty}{(t^i z_{r+1}\cdots z_s)_\infty}$$
$$\times \prod_{r=1}^{n-1} \prod_{y\in Y} \frac{(z_{r+1}\cdots z_{n-1} y/a_r)_\infty}{(z_{r+1}\cdots z_{n-1} y)_\infty} \prod_{x\in X}\prod_{y\in Y} \frac{(t z_1\cdots z_{n-1} xy)_\infty}{(z_1\cdots z_{n-1}xy)_\infty}.$$

Theorem 3.16 (A_n Cauchy-type formula II)**.** *Let $k_1 \leqslant \cdots \leqslant k_n$ be nonnegative integers. Fix $a_r = t^{k_r-k_{r+1}}$ for $1\leqslant r\leqslant n-1$. Then* (3.4.8) *also holds with $X^{(1)} = \{x_1,\dots,x_{k_1}\}$ and $Y^{(n)} = \{y_1,y_2,\dots\}$.*

Remark. Note that in the above theorems we adopt the convention that $\{y_1,\dots y_k\}$ with $k=\infty$ stands for the countably infinite alphabet $\{y_1,y_2,\dots\}$. In this case we also define $t^k := 0$.

Theorem 3.16 with $Y\mapsto (1-a)/(1-t)$ reduces to the A_n q-binomial theorem (3.4.1). Of course the infinite case of Theorem 3.15 may also be applied. In this case we pick up the free parameter a_{n-1} which gives another slight generalisation of the A_n q-binomial theorem.

Also observe that by Lemma 3.14 the summand of (3.4.8) vanishes unless the conditions (3.4.2) hold. These conditions are too restrictive in the case of Theorem 3.15 however, as a_{n-1} is a free parameter. Therefore the summand does not necessarily vanish for

$$\lambda_i^{(n-1)} < \lambda_{i-k_{n-1}+k_n}^{(n)}$$

in this case.

Proof of Theorems 3.15 and 3.16. Our goal is to eliminate the double products in the sum. First rewrite Corollary 3.12 with

$$(\lambda,\mu,\nu,k,\ell) \mapsto (\lambda^{(r)},\lambda^{(r+1)},\nu^{(r)},k_r,k_{r+1}) \tag{3.4.9}$$

so that for $\lambda^{(r)},\lambda^{(r+1)}$ partitions with $\lambda^{(r)}\leqslant k_r\leqslant k_{r+1}$ there holds

$$\sum_{\nu^{(r)}} t^{-|\nu^{(r)}|} P_{\lambda^{(r+1)}/\nu^{(r)}}\left[\frac{1-t^{k_{r+1}-k_r}}{1-t}\right] Q_{\lambda^{(r)}/\nu^{(r)}}\left[\frac{1-qt^{k_r-k_{r+1}-1}}{1-t}\right]$$
$$= t^{-k_r|\lambda^{(r+1)}|} P_{\lambda^{(r+1)}}\left[\frac{1-t^{k_{r+1}}}{1-t}\right] Q_{\lambda^{(r)}}\left[\frac{1-qt^{k_r-1}}{1-t}\right]$$
$$\times \prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \frac{(qt^{k_r-k_{r+1}+j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(qt^{k_r-k_{r+1}+j-i})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}.$$

Define $a_r := t^{k_r - k_{r+1}}$. Then this may be written as

$$\sum_{\nu^{(r)}} P_{\lambda^{(r+1)}/\nu^{(r)}}\left[\frac{1-a_r^{-1}}{1-t}\right] Q_{\lambda^{(r)}/\nu^{(r)}}\left[\frac{t-a_r q}{1-t}\right] \tag{3.4.10}$$
$$= P_{\lambda^{(r+1)}}\left[\frac{t^{-k_r}-a_r^{-1}}{1-t}\right] Q_{\lambda^{(r)}}\left[\frac{t-a_r q t^{k_{r+1}}}{1-t}\right] \prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \frac{(a_r q t^{j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(a_r q t^{j-i})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}.$$

We now repeat this process using Lemma 3.11. Making the substitutions (3.4.9) we obtain

$$\sum_{\nu^{(r)}} P_{\lambda^{(r+1)}/\nu^{(r)}}\left[\frac{1-a}{1-t}\right] Q_{\lambda^{(r)}/\nu^{(r)}}\left[\frac{t-q/a}{1-t}\right] \tag{3.4.11}$$
$$= P_{\lambda^{(r+1)}}\left[\frac{t^{-k_r}-a}{1-t}\right] Q_{\lambda^{(r)}}\left[\frac{t-q t^{k_{r+1}}/a}{1-t}\right] \prod_{i=1}^{k}\prod_{j=1}^{k_{r+1}} \frac{(q t^{j-i-1}/a)_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(q t^{j-i}/a)_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}.$$

If we replace $a \mapsto 1/a$ in the above and set $a = a_r$ then we see that this is simply (3.4.10). Hence there are two ways to interpret (3.4.10):

(1) (3.4.10) holds for partitions $\lambda^{(r)}, \lambda^{(r+1)}$ with $\ell(\lambda^{(r)}) \leqslant k_r \leqslant k_{r+1}$ and $a_r = t^{k_r - k_{r+1}}$,

(2) (3.4.10) holds for partitions $\lambda^{(r)}, \lambda^{(r+1)}$ with $\ell(\lambda^{(r)}) \leqslant k_r \in \mathbb{N}$, $\lambda^{(r+1)} \leqslant k_{r+1} \in \mathbb{N} \cup \{\infty\}$, and a_r an indeterminate.

In order to apply these results, note that the left-hand side of (3.4.8) vanishes unless $\ell(\lambda^{(1)}) \leqslant k_1$ as X has cardinality k_1. Furthermore for $2 \leqslant r \leqslant n-1$ the alphabet

$$X^{(r)} = t^{-k_{r-1}} \frac{1-t^{k_r}}{1-t}$$

has cardinality k_r, and so the summand vanishes unless $\ell(\lambda^{(r)}) \leqslant k_r$ for $1 \leqslant r \leqslant n-1$. We also assume that $k_1 \leqslant \cdots \leqslant k_{n-1}$ and $a_r := t^{k_r - k_{r+1}}$ for $1 \leqslant r \leqslant n-2$. Therefore we may use the interpretation (1) above to eliminate the first $n-2$ double products. For the last double product we may apply either interpretation. In the case of Theorem 3.15 we wish to use interpretation (2) in order to keep a_{n-1} as a free parameter. For the case of Theorem 3.16 we may use interpretation (1) to obtain the desired result. Denote the left-hand side of (3.4.8) by LHS(3.4.8). In either case we see that

$$\text{LHS}(3.4.8) = \sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \sum_{\nu^{(1)},\dots,\nu^{(n-1)}} P_{\lambda^{(1)}}(X^{(1)}) Q_{\lambda^{(n)}}(Y^{(n)})$$
$$\times \prod_{r=1}^{n-1} z_r^{|\lambda^{(r)}|} Q_{\lambda^{(r)}/\nu^{(r)}}\left[\frac{t-a_r q}{1-t}\right] P_{\lambda^{(r+1)}/\nu^{(r)}}\left[\frac{1-a_r^{-1}}{1-t}\right].$$

Interchanging the sums gives

$$\begin{aligned}\mathrm{LHS}(3.4.8) = \sum_{\nu^{(1)},\dots,\nu^{(n-1)}} & \bigg\{\bigg(\sum_{\lambda^{(1)}} P_{\lambda^{(1)}}(X^{(1)}) Q_{\lambda^{(1)}/\nu^{(1)}}\bigg[\frac{t-a_1 q}{1-t}\bigg]\bigg) \\ & \times \bigg(\sum_{\lambda^{(n)}} P_{\lambda^{(n)}/\nu^{(n-1)}}\bigg[\frac{1-a_{n-1}^{-1}}{1-t}\bigg] Q_{\lambda^{(n)}}(Y)\bigg) \\ & \times \prod_{r=2}^{n-1}\bigg(\sum_{\lambda^{(r)}} z_r^{|\lambda^{(r)}|} P_{\lambda^{(r)}/\nu^{(r-1)}}\bigg[\frac{1-a_{r-1}^{-1}}{1-t}\bigg] Q_{\lambda^{(r)}/\nu^{(r)}}\bigg[\frac{t-a_r q}{1-t}\bigg]\bigg)\bigg\}.\end{aligned}$$

With some additional notation we may write this in a simpler fashion. First define the alphabets

$$X^{(1)} = X^{(1,0)}, \qquad X^{(r,0)} = \frac{1-a_{r-1}^{-1}}{1-t} \tag{3.4.12}$$

for $2 \leqslant r \leqslant n$, and

$$Y^{(n)} = Y^{(n)}, \qquad Y^{(r)} = z_r \frac{t-a_r q}{1-t} \tag{3.4.13}$$

for $1 \leqslant r \leqslant n-1$. We then have the simpler

$$\mathrm{LHS}(3.4.8) = \sum_{\nu^{(1)},\dots,\nu^{(r-1)}} \prod_{r=1}^{n} \bigg(z_r^{|\nu^{(r)}|} \sum_{\lambda^{(r)}} P_{\lambda^{(r)}/\nu^{(r-1)}}[X^{(r,0)}] Q_{\lambda^{(r)}/\nu^{(r)}}[Y^{(r)}]\bigg), \tag{3.4.14}$$

where $z_n := 1$ and $\nu^{(0)} = \nu^{(n)} := 0$. We are now in a position to prove the following lemma.

Lemma 3.17. *For $0 \leqslant N \leqslant n-1$ there holds*

$$\begin{aligned}\mathrm{LHS}(3.4.8) = & \prod_{s=1}^{N} \prod_{x\in X^{(1,0)}} \frac{(a_s q z_1\cdots z_s x)_\infty}{(t z_1 \cdots z_s x)_\infty} \prod_{1\leqslant r<s\leqslant N} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1}\cdots z_s)_\infty}{(t^i z_{r+1}\cdots z_s)_\infty} \\ & \times \sum_{\nu^{(N+1)},\dots,\nu^{(n-1)}} \prod_{r=N+1}^{n} \bigg(z_r^{|\nu^{(r)}|} \sum_{\lambda^{(r)}} P_{\lambda^{(r)}/\nu^{(r-1)}}[X^{(r,N)}] Q_{\lambda^{(r)}/\nu^{(r)}}[Y^{(r)}]\bigg),\end{aligned}$$

where $z_n := 1, \nu^{(N)} = \nu^{(n)} := 0$, the alphabets $Y^{(r)}$ are given by (3.4.13)*, and*

$$\begin{aligned} X^{(N+1,N)} &:= (z_1\cdots z_N) X^{(1,0)} + \sum_{r=1}^{N} (z_{r+1}\cdots z_N)\frac{1-a_r^{-1}}{1-t} \\ X^{(r,N)} &:= \frac{1-a_{r-1}^{-1}}{1-t}, \qquad \text{for } N+2 \leqslant r \leqslant n.\end{aligned}$$

Proof. We proceed by induction. The base case $N = 0$ is the identity (3.4.14). Assume the lemma is true for some N. In order to prove it true for $N+1$, we require the formula

$$\sum_{\lambda} P_\lambda(Y) Q_{\lambda/\nu}\left[\frac{a-b}{1-t}\right] = P_\nu(Y) \prod_{y\in Y} \frac{(by)_\infty}{(ay)_\infty}. \tag{3.4.15}$$

Here, Y is an arbitrary countable alphabet and a, b are free parameters. To see this is true take (3.3.4) and make the plethystic substitution $Y \mapsto (a-b)/(1-t)$ to yield

$$\sum_{\lambda} P_\lambda(Y) Q_{\lambda/\nu}\left[\frac{a-b}{1-t}\right] = P_\nu(Y)\sigma\left[Y\frac{a-b}{1-q}\right].$$

An application of Lemma 2.10 in order to write the Cauchy kernel in terms of q-shifted factorials shows (3.4.15) is true. As $\nu^{(N)} := 0$ the sum over $\lambda^{(N+1)}$ is of the form (3.4.15), provided $0 \leqslant N \leqslant n-2$. Here the parameters become

$$(Y, a, b, \nu) \mapsto (X^{(N+1,N)}, tz_{N+1}, a_{N+1}qz_{N+1}, \nu^{(N+1)}).$$

For $0 \leqslant N \leqslant n-2$ the alphabet $X^{(N+1,N)}$ has finite cardinality as $a_r^{-1} = t^{k_{r+1}-k_r}$ in this case. It in fact has cardinality

$$|X^{(N+1,N)}| = k_1 + \sum_{r=1}^{N} k_{r+1} - k_r = k_{N+1}.$$

We now write $w_i := z_{i+1}\cdots z_N$, and note that $w_N := 1$. Then the alphabet $X^{(N+1,N)}$ may explicitly be written as

$$X^{(N+1,N)} = \{w_0x_1, \dots, w_0x_k\} \cup \bigcup_{r=1}^{N} \{w_rt^0, w_rt^1, \dots, w_rt^{k_{r+1}-k_r}\}.$$

This implies, by (3.4.15), that

$$\begin{aligned}
\mathrm{LHS}(3.4.8) = &\prod_{s=1}^{N+1} \prod_{x\in X^{(1,0)}} \frac{(a_sqz_1\cdots z_sx)_\infty}{(tz_1\cdots z_sx)_\infty} \\
&\times \prod_{1\leqslant r<s\leqslant N+1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_sqt^{i-1}z_{r+1}\cdots z_s)_\infty}{(t^iz_{r+1}\cdots z_s)_\infty} \\
&\times \sum_{\nu^{(N+1)},\dots,\nu^{(n-1)}} \Bigg[P_{\nu^{(N+1)}}[z_{N+1}X^{(N+1,N)}] \\
&\times \prod_{r=N+2}^{n}\Bigg(z_r^{|\nu^{(r)}|}\sum_{\lambda^{(r)}} P_{\lambda^{(r)}/\nu^{(r-1)}}[X^{(r,N)}]Q_{\lambda^{(r)}/\nu^{(r)}}[Y^{(r)}]\Bigg)\Bigg].
\end{aligned}$$

We may now carry out the sum over $\nu^{(N+1)}$ by (3.3.6). The result is

$$\sum_{\nu^{(N+1)}} P_{\nu^{(N+1)}}[z_{N+1}X^{(N+1,N)}]P_{\lambda^{(N+2)}/\nu^{(N+1)}}[X^{(N+2,N)}] \\ = P_{\lambda^{(N+2)}}[z_{N+1}X^{(N+1,N)} + X^{(N+2,N)}].$$

However it is clear from our definition that

$$z_{N+1}X^{(N+1,N)} + X^{(N+2,N)} = X^{(N+2,N+1)}.$$

Therefore we have the formula

$$\begin{aligned}\mathrm{LHS}(3.4.8) = {} & \prod_{s=1}^{N+1} \prod_{x\in X^{(1,0)}} \frac{(a_s q z_1 \cdots z_s x)_\infty}{(t z_1 \cdots z_s x)_\infty} \\ & \times \prod_{1\leqslant r<s\leqslant N+1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1}\cdots z_s)_\infty}{(t^i z_{r+1}\cdots z_s)_\infty} \\ & \times \sum_{\nu^{(N+2)},\dots,\nu^{(n-1)}} \prod_{r=N+2}^{n} \biggl(z_r^{|\nu^{(r)}|} \sum_{\lambda^{(r)}} P_{\lambda^{(r)}/\nu^{(r-1)}}[X^{(r,N)}] \\ & \qquad \times Q_{\lambda^{(r)}/\nu^{(r)}}[Y^{(r)}] \biggr).\end{aligned}$$

Here we define $\nu^{(N+1)} := 0$, so that this is simply Lemma 3.17 with $N+1$ in place of N. Hence the result is true by induction. ■

With the lemma established we may set $N = n-1$ yielding

$$\begin{aligned}\mathrm{LHS}(3.4.8) = \prod_{s=1}^{n-1} \frac{(a_s q z_1 \cdots z_s x)_\infty}{(t z_1 \cdots z_s x)_\infty} \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(asqt^{i-1} z_{r+1}\cdots z_s)_\infty}{(t^i z_{r+1}\cdots z_s)_\infty} \\ \times \sum_{\lambda^{(n)}} P_{\lambda^{(n)}}[X^{(n,n-1)}]Q_{\lambda^{(n)}}(Y^{(1)}),\end{aligned}$$

where we recall that

$$X^{(n,n-1)} = (z_1\cdots z_{n-1})X + \sum_{r=1}^{n-1} (z_{r+1}\cdots z_{n-1}) \frac{1-a_r^{-1}}{1-t}.$$

The Cauchy identity (3.1.15) may be used to evaluate this last sum. However in the case of Theorem 3.15 we require the plethystically substituted identity

$$\sum_\lambda P_\lambda\Bigl[X + \frac{a-b}{1-t}\Bigr] Q_\lambda(Y) = \prod_{y\in Y} \frac{(by)_\infty}{(ay)_\infty} \prod_{x\in X}\prod_{y\in Y} \frac{(txy)_\infty}{(xy)_\infty} \tag{3.4.16}$$

as the alphabet $X^{(n,n-1)}$ contains the term $\frac{1-a_{n-1}^{-1}}{1-t}$ for a parameter a_{n-1}. This is a simple consequence of the original Cauchy identity and Lemma 2.10. We now take (3.4.16) with

$$X \mapsto (z_1 \cdots z_{n-1})X^{(n,n-1)} + \sum_{r=1}^{n-2}(z_{r+1}\cdots z_{n-1})\frac{1-a_r^{-1}}{1-t},$$

and

$$(a,b) \mapsto (z_1 \cdots z_{n-1}, z_1 \cdots z_{n-1}/a_{n-1}).$$

An application of this yields the formula

$$\begin{aligned}\mathrm{LHS}(3.4.8) = &\prod_{s=1}^{n-1}\prod_{x\in X^{(0,1)}}\frac{(a_s q z_1\cdots z_s x)_\infty}{(tz_1\cdots z_s x)_\infty}\prod_{1\leqslant r<s\leqslant n-1}\prod_{i=1}^{k_{r+1}-k_r}\frac{(a_s q t^{i-1} z_{r+1}\cdots z_s)_\infty}{(t^i z_{r+1}\cdots z_s)_\infty}\\ &\times\prod_{r=1}^{n-1}\prod_{y\in Y}\frac{(z_{r+1}\cdots z_{n-1}y/a_r)_\infty}{(z_{r+1}\cdots z_{n-1}y)_\infty}\prod_{x\in X}\prod_{y\in Y}\frac{(tz_1\cdots z_{n-1}xy)_\infty}{(z_1\cdots z_{n-1}xy)_\infty}.\end{aligned}$$

In the case of Theorem 3.16 the standard Cauchy identity would suffice as $a_{n-1} := t^{k_{n-1}-k_n}$, which in turn means $X^{(n,n-1)}$ is finite and contains no free parameters. Doing this yields (notationally) the same result. ■

Remark.

1. In both of the above theorems the parameter z_1 is redundant. Indeed for $n \geqslant 2$ if we scale $X \mapsto X/z_1$ then any reference to z_1 is removed.

2. When $n = 2$ we may write Theorem 3.15 in a simpler fashion. Let $X = \{x_1,\dots,x_k\}$ and $Y = \{y_1,\dots,y_\ell\}$ be such that $k \in \mathbb{N}$ and $\ell \in \mathbb{N}\cup\{\infty\}$. Then we have

$$\begin{aligned}\sum_{\lambda,\mu} t^{|\lambda|-k|\mu|}P_\lambda(X)Q_\mu(Y)P_\mu\Big[\frac{1-t^k/a}{1-t}\Big]Q_\lambda\Big[\frac{1-aqt^{\ell-1}}{1-t}\Big]&\\ \times\prod_{i=1}^{k}\prod_{j=1}^{\ell}\frac{(aqt^{j-i-1})_{\lambda_i-\mu_j}}{(aqt^{j-i})_{\lambda_i-\mu_j}}&\\ =\prod_{x\in X}\frac{(aqx)_\infty}{(tx)_\infty}\prod_{y\in Y}\frac{(y/a)_\infty}{(y)_\infty}\prod_{x\in X}\prod_{y\in Y}\frac{(txy)_\infty}{(xy)_\infty}.&\end{aligned}$$

Here we have eliminated z_1 as above. For $\ell \in \mathbb{N}$ this is [War10, Theorem 1.2] (with $a \mapsto a/q$).

CHAPTER 4

q-INTEGRALS

The Selberg integral is an important example of a hypergeometric integral. Here we motivate hypergeometric integrals, and explain the connection between q-Selberg integrals and q-binomial theorems. As an extension of this we prove a q-analogue of the Alba–Fateev–Litvinov–Tarnopolskiy integral using the Cauchy identity for Macdonald polynomials.

4.1 Hypergeometric Integrals

Recall that for complex parameters α and β with strictly positive real parts the beta integral is given by [Eul30]

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{4.1.1}$$

As mentioned in the introduction, the Selberg integral is a generalisation of the beta integral, and both of these integrals are referred to as hypergeometric integrals. This is due to the relationship between the beta integral and hypergeometric series. In general a series $\sum c_n$ is called *hypergeometric* if the ratio of consecutive terms c_{n+1}/c_n is a rational function of n. Perhaps the simplest example of a hypergeometric series is the Taylor series of the exponential function, given by

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Of course any geometric series is also a hypergeometric series. Indeed many of the well-known functions of classical analysis have representations in terms of hypergeometric series. The most important of the standard hypergeometric series is Gauss' hypergeometric function. This is defined for $|z| < 1$ by the series representation

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{4.1.2}$$

and elsewhere by analytic continuation. Here $(x)_n$ is the classical Pochhammer symbol, defined by $(x)_0 := 1$ and for n a positive integer by

$$(x)_n := (x)(x+1)\cdots(x+n-1). \tag{4.1.3}$$

The function ${}_2F_1(z)$ is a specific case of the more general hypergeometric series ${}_rF_s(z)$, for more details see [AAR99, p. 62]. The relationship between

(4.1.2) and the beta integral originates from Euler's integral representation of the hypergeometric function. If $\mathrm{Re}(b), \mathrm{Re}(c) > 0$ then

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t, \tag{4.1.4} $$

defined in the z-plane with a branch cut from 1 to ∞ [AAR99, Theorem 2.2.1]. Hence the integral representations of the beta and gamma functions are dubbed hypergeometric integrals.

Our method for proving hypergeometric integral evaluations in this thesis relies on the notion of q-integrals. As with the relationship between hypergeometric series and hypergeometric integrals, there is a relationship between q-integrals and generalised hypergeometric series. There are in fact many generalisations of hypergeometric series, the most important of which, for our purposes, are basic hypergeometric series. These involve an extra parameter q, and require that the ratio of consecutive terms is a rational function of q^n. In order to write this succinctly we introduce the standard condensed notation

$$ (a_1; q)_n \cdots (a_k; q)_n = (a_1, \dots, a_k; q)_n $$

for $n \in \mathbb{N} \cup \{\infty\}$. The corresponding generalisation of the hypergeometric function (4.1.2) lies in the more general series [GR90, p. 32]

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n, \tag{4.1.5} $$

where q and z are complex numbers such that $|q|, |z| < 1$ and r is a nonnegative integer. When dealing with basic hypergeometric series we require that q lies within the open unit disk to ensure convergence is not a problem. Taking the limit $q \to 1^-$ in (4.1.5) returns the hypergeometric series. This may be generalised to ${}_r\phi_s$ where $s \neq r+1$, however we only require series of the form ${}_{r+1}\phi_r$. When $r = 0$ the basic hypergeometric series (4.1.5) may in fact be summed to give a closed form expression.

Theorem 4.1 (q-binomial Theorem)**.** *For q and z complex numbers such that $|q|, |z| < 1$ we have that*

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}. \tag{4.1.6} $$

Proof. Fix the parameters a and q such that $|q| < 1$. Then the infinite product on the right of (4.1.6) is absolutely and uniformly convergent for all $|z| \leqslant 1-\varepsilon$ with $\varepsilon > 0$. Therefore it defines an analytic function inside the open complex unit disk. Taking the Taylor series expansion gives

$$ \frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} c_n z^n. $$

We wish to find the coefficients c_n. Observe that the infinite product, as a function of z, satisfies the relation

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \frac{(1-az)}{(1-z)}\frac{(aqz;q)_\infty}{(qz;q)_\infty}.$$

The Taylor series expansion must also satisfy this identity. Therefore we have that

$$(1-z)\sum_{n=0}^{\infty} c_n z^n = (1-az)\sum_{n=0}^{\infty} c_n q^n z^n.$$

This is equivalent to

$$\sum_{n=0}^{\infty} c_n(z^n - z^{n+1}) = \sum_{n=0}^{\infty} c_n q^n (z^n - az^{n+1}).$$

For $n \geqslant 0$ we may equate the coefficients of z^n on both sides to obtain the recursion

$$c_n = c_{n-1}\frac{1-aq^{n-1}}{1-q^n}.$$

Upon iteration this shows us that

$$c_n = \frac{(a;q)_n}{(q;q)_n}.$$

This completes the proof. ∎

Remark. The above proof has an analytic flavour, in particular requiring results concerning the convergence of infinite products. For a different proof using q-difference operators, as well as the above proof, see [AAR99, Theorem 10.2.1].

Before seeing how the q-binomial theorem relates to q-integrals, we first need a transformation formula for the ${}_2\phi_1$ series, due to Heine [And86, §2.1].

Lemma 4.2. *For nonzero complex b, q and z such that $|b|, |q|, |z| < 1$,*

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b, z \\ az \end{matrix}; q, b\right]. \tag{4.1.7}$$

Proof. This is a simple application of Theorem 4.1. Indeed

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty} z^n.$$

Applying the q-binomial theorem to the term $(cq^n;q)_\infty/(bq^n;q)_\infty$ gives

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n} z^n \sum_{m=0}^{\infty}\frac{(c/b;q)_\infty}{(q;q)_\infty}(bq^n)^m.$$

Another application of the q-binomial theorem yields

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m (azq^m;q)_m}{(q;q)_m (zq^m;q)_m} b^m $$

Rewriting the right hand side shows that

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] &= \frac{(b, az;q)_\infty}{(c, z;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b, z;q)_m}{(q, az;q)_m} b^m \\ &= \frac{(b, az;q)_\infty}{(c, z;q)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, z \\ az \end{matrix}; q, b\right]. \end{aligned} $$ ∎

This identity may be interpreted as a q-analogue of Euler's integral formula (4.1.4), however now involving q-integrals and q-gamma functions. Recall that for $z \in \mathbb{C}$ the q-gamma function is given by

$$ \Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}. \tag{4.1.8} $$

Note that this has poles at the nonpositive integers just as with the classical gamma function. One also has that

$$ \lim_{q \to 1^-} \Gamma_q(z) = \Gamma(z). $$

This result admits a simple proof due to Gosper in the case of a formal limit [And86, Appendix A]. For a more rigorous analytic proof see [Koo90, Appendix B]. The definition of the q-integral is slightly more involved. Here we restrict q further, allowing only $q \in (0, 1)$. Then the q-integral or Jackson-integral over $[0, 1]$ is given by

$$ \int_0^1 f(t)\, \mathrm{d}_q t = (1-q) \sum_{n=0}^{\infty} q^n f(q^n). \tag{4.1.9} $$

This is defined for any function $f(t)$ on $[0, 1]$ such that the sum on the right hand side converges. The notation $\mathrm{d}_q t$ indicates when the integral is a q-integral rather than a Riemann integral. This is referred to as the Fermat measure by some authors [AAR99, §10.1]. For a continuous function the $q \to 1^-$ limit of (4.1.9) returns the Riemann integral on $[0, 1]$. With these definitions established we may rewrite Lemma 4.2 as an identity involving q-integrals. To do so, set $a = q^\alpha$, $b = q^\beta$ and $c = q^\gamma$. Then (4.1.7) becomes

$$ {}_2\phi_1\left[\begin{matrix} q^\alpha, q^\beta \\ q^\gamma \end{matrix}; q, z\right] = \frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(qt;q)_{\gamma-\beta-1}}{(zt;q)_\alpha} \mathrm{d}_q t. \tag{4.1.10} $$

It is well-known that one may derive the beta integral from Euler's integral representation of the hypergeometric function. In an almost identical way one may obtain a q-analogue of the beta integral from the equation (4.1.10). However as in the case of the beta integral, this is unnecessary and a q-analogue of (4.1.1) is in fact equivalent to the q-binomial theorem (Theorem 4.1). To see this first consider

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Then as with rewriting the ${}_2\phi_1$ transform we set $a = q^\alpha$ and $z = q^\beta$. Some elementary manipulations using q-shifted factorials gives the identity

$$\int_0^1 t^{\alpha-1}(qt;q)_{\beta-1}\,\mathrm{d}_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}. \tag{4.1.11}$$

Thanks to the definitions of the q-integral and q-gamma function it is clear that sending $q \to 1^-$ in (4.1.11) the original beta integral is obtained. It should be noted that this is but one extension of the beta integral, and there are many other q-beta integrals over different domains and in both discrete and continuous forms. Much of this work is due to Askey [Ask80a] and Andrews and Askey [AA81]. These results are also covered in [AAR99, §10.8] and [GR90].

4.2 The Askey–Habsieger–Kadell integral

So far we have seen that some hypergeometric integrals admit q-analogues, and that these q-integrals are equivalent to identities coming from basic hypergeometric series. In this section we aim to extend this to the Selberg integral, and its most important q-analogue, the Askey–Habsieger–Kadell integral.

Fix a positive integer k. Recall the Selberg integral:

$$\begin{aligned} S_k(\alpha,\beta,\gamma) &:= \int\limits_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i - t_j|^{2\gamma}\,\mathrm{d}t \\ &= \prod_{j=1}^{k} \frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(k+j-2)\gamma)\Gamma(1+\gamma)}, \end{aligned} \tag{4.2.1}$$

where $\alpha,\beta,\gamma \in \mathbb{C}$ satisfying $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ and

$$\mathrm{Re}(\gamma) > -\min\{1/k, \mathrm{Re}(\alpha)/(k-1), \mathrm{Re}(\beta)/(k-1)\}.$$

Setting $k = 1$ recovers the beta integral (4.1.1), and hence the Selberg integral is also considered a hypergeometric integral. It is therefore natural to ask if there is a corresponding basic hypergeometric series that, when specialised, gives a q-analogue of (4.2.1). The answer to this question involves generalised basic hypergeometric series with Macdonald polynomial argument, originally introduced independently by Kaneko and Macdonald [Kan96, Definition 3.4], [Mac, §1 (1.10)]. These series replace the term z^n in the sum by a Macdonald polynomial indexed by an alphabet X. Doing this for the ${}_{r+1}\phi_r$ series gives

$$ {}_{r+1}\Phi_r\left[\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix};q,t;X\right] := \sum_\lambda t^{n(\lambda)} \frac{P_\lambda(X;q,t)}{c'_\lambda(q,t)} \frac{(a_1,\dots,a_{r+1};q,t)_\lambda}{(b_1,\dots,b_r;q,t)_\lambda}. \tag{4.2.2} $$

Here we have used the notations of Chapter 3 with regards to Macdonald polynomials. Note that we view this sum as a formal one, however it is not hard to impose appropriate restrictions when the alphabet X is finite in order to view the above series as a hypergeometric function [Kan96].

As in the beta integral case, we are interested in summing the series when $r = 0$. This gives the following q-binomial theorem for Macdonald polynomials, obtained independently by Kaneko and Macdonald in [Kan96, Theorem 3.5] and [Mac, p. 4].

Theorem 4.3 (Kaneko–Macdonald q-binomial Theorem)**.** *For an arbitrary alphabet X there holds*

$$ \sum_\lambda \frac{t^{n(\lambda)}(a;q,t)_\lambda}{c'_\lambda(q,t)} P_\lambda(X;q,t) = \prod_{x\in X} \frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{4.2.3} $$

where the sum on the left is over all partitions.

Proof. This is in fact specialised case of the Cauchy identity for Macdonald polynomials (3.1.15), which states that for alphabets X and Y,

$$ \sum_\lambda P_\lambda(X;q,t)Q_\lambda(Y;q,t) = \prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty}. $$

We make a plethystic substitution for the alphabet Y. Let a be a parameter and set $Y \mapsto (1-a)/(1-t)$. Then by the specialisation formula (3.2.20) we have

$$ Q_\lambda\left(\left[\frac{1-a}{1-t}\right];q,t\right) = \frac{t^{n(\lambda)}(a;q,t)_\lambda}{c'_\lambda(q,t)}. $$

Recall that the Cauchy kernel on the right hand side of the Cauchy identity may be written as

$$ \sigma\left[XY\frac{1-t}{1-q}\right] = \prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty}. $$

Setting $Y \mapsto (1-a)/(1-t)$ here leaves the expression

$$ \sigma\left[X\frac{1-a}{1-q}\right] = \prod_{x\in X} \frac{(ax;q)_\infty}{(x;q)_\infty}. $$

Putting this together we see that

$$ \sum_\lambda \frac{t^{n(\lambda)}(a;q,t)_\lambda}{c'_\lambda(q,t)} P_\lambda(X;q,t) = \prod_{x\in X} \frac{(ax;q)_\infty}{(x;q)_\infty}. $$

■

Using this summation formula it is possible to prove a q-analogue of the Selberg integral, and then in the limit obtain the Selberg integral itself. First we require the multidimensional q-integral. This is given by

$$\int\limits_{[0,1]^k} f(t_1,\dots,t_k)\,\mathrm{d}_q t_1\cdots \mathrm{d}_q t_k = (1-q)^k \sum_{\sigma} f(q^{\sigma}) q^{|\sigma|}, \tag{4.2.4}$$

where the sum on the right is over all weak compositions σ, and we set $f(q^{\sigma}) := f(q^{\sigma_1},\dots,q^{\sigma_k})$. The particular q-Selberg integral we now prove was first conjectured and proved for low-dimensional cases by Askey, who also conjectured several other q-Selberg integrals [Ask80b]. Full proofs followed in the independent work of Habsieger [Hab88] and Kadell [Kad88a, Theorem 2].

Theorem 4.4 (Askey–Habsieger–Kadell integral)**.** *Let α,β be complex numbers such that* $\mathrm{Re}(\alpha)>0$ *and β is not a nonpositive integer. Then for positive integers γ and k there holds*

$$\begin{aligned}\int\limits_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(qt_i;q)_{\beta-1} \prod_{1\leqslant i<j\leqslant k} t_i^{2\gamma}(t_j q^{1-\gamma}/t_i;q)_{2\gamma}\,\mathrm{d}_q t \qquad\qquad (4.2.5)\\ = q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} \prod_{j=1}^{k} \frac{\Gamma_q(\alpha+(j-1)\gamma)\Gamma_q(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma_q(\alpha+\beta+(k+j-2)\gamma)\Gamma_q(1+\gamma)}.\end{aligned}$$

Proof. We make the substitution $X\mapsto b\langle 0\rangle_k$ for k a fixed positive integer in the Kaneko–Macdonald q-binomial theorem (4.2.3) to obtain

$$\sum_{\lambda} \frac{t^{n(\lambda)} b^{|\lambda|}(a;q,t)_{\lambda}}{c'_{\lambda}(q,t)} P_{\lambda}(\langle 0\rangle_k;q,t) = \prod_{i=1}^{k} \frac{(abt^{k-i};q)_{\infty}}{(bt^{k-i};q)_{\infty}}.$$

Now applying the principal specialisation formula for Macdonald polynomials (3.2.13) gives

$$\sum_{\lambda} \frac{t^{2n(\lambda)} b^{|\lambda|}(a,t^k;q,t)_{\lambda}}{c_{\lambda}(q,t)c'_{\lambda}(q,t)} = \prod_{i=1}^{k} \frac{(abt^{k-i};q)_{\infty}}{(bt^{k-i};q)_{\infty}}.$$

Expressing this in terms of q-shifted factorials by (3.2.11) and (3.2.12) we see that

$$\begin{aligned}\sum_{\lambda} t^{2n(\lambda)} b^{|\lambda|} \prod_{i=1}^{k} \frac{(at^{1-i};q)_{\lambda_i}}{(qt^{k-i};q)_{\lambda_i}} \prod_{1\leqslant i<j\leqslant k} \frac{(qt^{j-i},t^{j-i+1};q)_{\lambda_i-\lambda_j}}{(qt^{j-i-1},t^{j-i};q)_{\lambda_i-\lambda_j}} \qquad (4.2.6)\\ = \prod_{i=1}^{k} \frac{(abt^{k-i};q)_{\infty}}{(bt^{k-i};q)_{\infty}}.\end{aligned}$$

We now set the parameters according to $(a,b,t) \mapsto (q^{\beta+(k-1)\gamma}, q^{\alpha}, q^{\gamma})$. This yields

$$\sum_{\lambda} q^{2n(\lambda)\gamma+\alpha|\lambda|} \prod_{i=1}^{k} \frac{(q^{\beta+(k-i)\gamma};q)_{\lambda_i}}{(q^{1+(k-i)\gamma};q)_{\lambda_i}} \prod_{1\leqslant i<j\leqslant k} \frac{(q^{1+(j-i)\gamma}, q^{(j-i+1)\gamma};q)_{\lambda_i-\lambda_j}}{(q^{1+(j-i-1)\gamma}, q^{(j-i)\gamma};q)_{\lambda_i-\lambda_j}} = \prod_{i=1}^{k} \frac{(q^{\alpha+\beta+(2k-i-1)\gamma};q)_\infty}{(q^{\alpha+(k-i)\gamma};q)_\infty}.$$

Recalling the definitions

$$(a;q)_z = \frac{(a;q)_\infty}{(aq^z;q)_\infty}$$

from (3.2.10) and $n(\lambda) = \sum_{i\geqslant 1}(i-1)\lambda_i$ from (2.1.1), we obtain

$$\begin{aligned}\sum_{\lambda}\prod_{i=1}^{k} q^{2(i-1)\lambda_i+\alpha\lambda_i}(q^{1+(k-i)\gamma+\lambda_i};q)_{\beta-1} \\ \times \prod_{1\leqslant i<j\leqslant k}(1-q^{(j-i)\gamma+\lambda_i-\lambda_j})(q^{1+(j-i-1)\gamma+\lambda_i-\lambda_j};q)_{2\gamma-1} \\ = \prod_{i=1}^{k} \frac{(q^{1+(k-i)\gamma}, q^{\alpha+\beta+(2k-i-1)\gamma};q)_\infty}{(q^{\beta+(k-i)\gamma}, q^{\alpha+(k-i)\gamma};q)_\infty} \prod_{1\leqslant i<j\leqslant k} \frac{(q^{1+(j-i-1)\gamma}, q^{(j-i)\gamma};q)_\infty}{(q^{1+(j-i)\gamma}, q^{(j-i+1)\gamma};q)_\infty}.\end{aligned} \tag{4.2.7}$$

In order to write the left-hand side of this expression as a multiple q-integral we require the following lemma.

Lemma 4.5 ([War05, Lemma 3.1])**.** *Let $\alpha = (\alpha_1,\dots,\alpha_n)$ be a weak composition and γ a positive integer. Let f_α satisfy $f_\alpha = f_{w(\alpha)}$ for all $w \in \mathfrak{S}_n$ and $f_\alpha = 0$ for $0 \leqslant |\alpha_i - \alpha_{i+1}| \leqslant \gamma - 1$. Then we have the formal summation identity*

$$\sum_{\lambda} f_{\lambda+\gamma\delta^{(n)}} = \frac{(1-t)^n}{(t;t)_n}\sum_{\alpha} f_\alpha \prod_{1\leqslant i<j\leqslant n}\frac{q^{\alpha_i}-tq^{\alpha_j}}{q^{\alpha_i}-q^{\alpha_j}}$$

where the sum on the left is over all partitions λ such that $\ell(\lambda) \leqslant n$, and similarly for the sum over weak compositions. Here $\delta^{(n)} = (n-1, n-2, \dots, 1, 0)$ in the staircase partition and the notation $\lambda + \gamma\delta^{(n)}$ stands for the partition $(\lambda_1 + (n-1)\gamma, \lambda_2 + (n-2)\gamma, \dots, \lambda_n + 0\gamma)$.

Proof. Begin with the left-hand side of the identity. We rescale the summation indices by $\lambda_i \mapsto \lambda_i - \gamma(n-i)$. The summation now becomes $\sum'_\lambda f_\lambda$ where the prime indicates the additional restriction $\lambda_i - \lambda_{i+1} \geqslant \gamma$. We require this as the original sum is over all partitions, i.e., over sequences of nonnegative integers such that $\lambda_i - \lambda_{i+1} \geqslant 0$. This implies that $\lambda_i - \lambda_{i+1} \geqslant \gamma$ after the shift. However $f_\lambda = 0$ if $0 \leqslant \lambda_i - \lambda_j \leqslant \gamma - 1$ and so the prime indicating the restriction on the sum is unnecessary. The function f_λ is also symmetric in the variables, and vanishes if $\lambda_i = \lambda_j$. Hence we may replace the sum with a sum over all weak compositions, provided we divide through by $n!$. From this we obtain

$$\sum_{\lambda} f_\lambda = \frac{1}{n!}\sum_{\alpha} f_\alpha.$$

In order to procure the extra factors on the right-hand side we require the formula [Mac15, p. 207]

$$\sum_{w\in\mathfrak{S}_n} w\Big(\prod_{1\leqslant i<j\leqslant n}\frac{x_i-tx_j}{x_i-x_j}\Big)=\frac{(t;t)_n}{(1-t)^n}.$$

For our purposes we take $x_i := q^{\alpha_i}$. Rewriting our sum gives

$$\begin{aligned}\frac{1}{n!}\sum_{\alpha} f_\alpha &= \frac{1}{n!}\sum_{\alpha} f_\alpha \frac{(1-t)^n}{(t;t)_n}\sum_{w\in\mathfrak{S}_n} w\Big(\prod_{1\leqslant i<j\leqslant n}\frac{q^{\alpha_i}-tq^{\alpha_j}}{q^{\alpha_i}-q^{\alpha_j}}\Big)\\ &= \frac{(1-t)^n}{n!(t;t)_n}\sum_{\alpha}\sum_{w\in\mathfrak{S}_n} w\Big(f_\alpha\prod_{1\leqslant i<j\leqslant n}\frac{q^{\alpha_i}-tq^{\alpha_j}}{q^{\alpha_i}-q^{\alpha_j}}\Big)\\ &= \frac{(1-t)^n}{(t;t)_n}\sum_{\alpha} f_\alpha \prod_{1\leqslant i<j\leqslant n}\frac{q^{\alpha_i}-tq^{\alpha_j}}{q^{\alpha_i}-q^{\alpha_j}}.\end{aligned}$$

The last line follows as

$$f_\alpha\prod_{1\leqslant i<j\leqslant n}\frac{q^{\alpha_i}-tq^{\alpha_j}}{q^{\alpha_i}-q^{\alpha_j}}$$

is a symmetric function. Note that the product is a symmetric function as it is a ratio of two skew-symmetric functions. This completes our proof. ■

We now return to (4.2.7). In order to find the function f_λ in our case we rewrite the identity as

$$\begin{aligned}q^{-\alpha\gamma\binom{k}{2}-2\gamma^2\binom{k}{3}}\sum_{\lambda}\prod_{i=1}^{k} q^{2(i-1)\gamma(\lambda_i+(k-i)\gamma)+\alpha(\lambda_i+(k-i)\gamma)}(q^{1+(k-i)\gamma+\lambda_i};q)_{\beta-1}&\\ \times\prod_{1\leqslant i<j\leqslant k}(1-q^{(j-i)\gamma+\lambda_i-\lambda_j})(q^{1+(j-i-1)\gamma+\lambda_i-\lambda_j};q)_{2\gamma-1}&\\ =\prod_{i=1}^{k}\frac{(q^{1+(k-i)\gamma},q^{\alpha+\beta+(2k-i-1)\gamma};q)_\infty}{(q^{\beta+(k-i)\gamma},q^{\alpha+(k-i)\gamma};q)_\infty}\prod_{1\leqslant i<j\leqslant k}\frac{(q^{1+(j-i-1)\gamma},q^{(j-i)\gamma};q)_\infty}{(q^{1+(j-i)\gamma},q^{(j-i+1)\gamma};q)_\infty}.&\end{aligned}$$

We then see that f_λ is given by

$$f_\lambda = q^{\alpha|\lambda|}\prod_{i=1}^{k}(q^{1+\lambda_i};q)_{\beta-1}\prod_{1\leqslant i<j\leqslant k} q^{(2\gamma-1)\lambda_j}(q^{\lambda_j}-q^{\lambda_i})(q^{1-\gamma+\lambda_i-\lambda_j};q)_{2\gamma-1}.$$

To see that this is symmetric under permutation of the λ_i note that clearly the term $q^{\alpha|\lambda|}\prod_{i=1}^{k}(q^{1+\lambda_i};q)_{\beta-1}$ is symmetric. Now by virtue of the identity

$$(a;q)_n=(q^{1-n}/a;q)_n(-a)^n q^{\binom{n}{2}}$$

the term $(q^{1-\gamma+\lambda_i-\lambda_j};q)_{2\gamma-1}$ is skew-symmetric in λ. Hence as $q^{(2\gamma-1)\lambda_j}(q^{\lambda_j}-q^{\lambda_i})$ is also skew-symmetric in λ the entire expression is symmetric. Also note that if $|\lambda_i-\lambda_j|\in\{0,1,\dots,\gamma-1\}$ then the term $(q^{1-\gamma+\lambda_i-\lambda_j};q)_{2\gamma-1}$ will vanish. Hence the conditions of Lemma 4.5 are satisfied and we replace the sum over λ by a sum over σ, where σ stands for a weak composition:

$$\begin{aligned}
&q^{-\alpha\gamma\binom{k}{2}-2\gamma^2\binom{k}{3}}\frac{(1-q^\gamma)^k}{(q^\gamma;q^\gamma)_k}\sum_\sigma q^{\alpha|\sigma|}\prod_{i=1}^k(q^{1+\sigma_i};q)_{\beta-1}\\
&\qquad\times\prod_{1\leqslant i<j\leqslant k}q^{(2\gamma-1)\sigma_j}\frac{(q^{\sigma_j}-q^{\sigma_i})(q^{\sigma_i}-q^{\sigma_j+\gamma})}{(q^{\sigma_i}-q^{\sigma_j})}(q^{1-\gamma+\sigma_i-\sigma_j};q)_{2\gamma-1}\\
&=\prod_{i=1}^k\frac{(q^{1+(k-i)\gamma},q^{\alpha+\beta+(2k-i-1)\gamma};q)_\infty}{(q^{\beta+(k-i)\gamma},q^{\alpha+(k-i)\gamma};q)_\infty}\prod_{1\leqslant i<j\leqslant k}\frac{(q^{1+(j-i-1)\gamma},q^{(j-i)\gamma};q)_\infty}{(q^{1+(j-i)\gamma},q^{(j-i+1)\gamma};q)_\infty}.
\end{aligned}$$

If we move terms independent of the sum to the right-hand side and manipulate the left-hand side we obtain

$$\begin{aligned}
&\sum_\sigma q^{\alpha|\sigma|}\prod_{i=1}^k(q^{1+\sigma_i};q)_{\beta-1}\prod_{1\leqslant i<j\leqslant k}q^{2\gamma\sigma_i}(q^{1-\gamma+\sigma_j-\sigma_i};q)_{2\gamma}\\
&\quad=q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}}\frac{(q^\gamma;q^\gamma)_k}{(1-q^\gamma)^k}\prod_{i=1}^k\frac{(q^{1+(k-i)\gamma},q^{\alpha+\beta+(2k-i-1)\gamma};q)_\infty}{(q^{\beta+(k-i)\gamma},q^{\alpha+(k-i)\gamma};q)_\infty}\\
&\qquad\times\prod_{1\leqslant i<j\leqslant k}\frac{(q^{1+(j-i-1)\gamma},q^{(j-i)\gamma};q)_\infty}{(q^{1+(j-i)\gamma},q^{(j-i+1)\gamma};q)_\infty}.
\end{aligned}$$

Comparing this with the right-hand side of (4.2.4) we see that our left-hand side is precisely a q-integral but without the factor $(1-q)^k$. To remedy this we multiply both sides by $(1-q)^k$ and apply

$$\frac{\Gamma_q(z+n)}{\Gamma_q(z)}=\frac{(q^z;q)_n}{(1-q)^n}$$

in order to rewrite the right-hand side in terms of q-gamma functions. This yields

$$\begin{aligned}
&\int_{[0,1]^k}\prod_{i=1}^k t_i^{\alpha-1}(qt_i;q)_{\beta-1}\prod_{1\leqslant i<j\leqslant k}t_i^{2\gamma}(t_jq^{1-\gamma}/t_i;q)_{2\gamma}\,\mathrm{d}_qt_1\cdots\mathrm{d}_qt_k\\
&\quad=q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}}\prod_{j=1}^k\frac{\Gamma_q(\alpha+(j-1)\gamma)\Gamma_q(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma_q(\alpha+\beta+(k+j-2)\gamma)\Gamma_q(1+\gamma)}.
\end{aligned}$$

This is precisely (4.2.5). The restriction on β comes from the q-gamma function $\Gamma_q(\beta)$ occurring on the right-hand side of the integral. The restriction on α comes from the fact that the sum form of the q-integral is required to be absolutely convergent. ■

With the above integral established all that remains is to take the limit as $q \to 1^-$ and one obtains the Selberg integral for γ a positive integer. The restriction on γ comes from Lemma 4.5, which is only necessary to explicitly obtain the Askey–Habsieger–Kadell integral. However the expression (4.2.7) is still a valid q-integral, but on the restricted domain $0 < t_1 < \cdots < t_k < 1$. In the limit this becomes the full Selberg integral for general γ. Note that in order to take the limit we require $\mathrm{Re}(\beta) > 0$ to ensure convergence. This further emphasises that the symmetry in α and β in the Selberg integral is not present at the q-level.

4.3 A q-analogue of the Alba–Fateev–Litvinov–Tarnopolskiy integral

We now turn our attention to the integral of principal interest to this thesis, the Alba–Fateev–Litvinov–Tarnopolskiy (AFLT) integral. This states that

$$\begin{aligned}\int_{[0,1]^k} & P_\mu^{1/\gamma}(t) P_\nu^{(1/\gamma)}[t+\beta/\gamma-1] \\ & \times \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i-t_j|^{2k}\, \mathrm{d}t_1\cdots \mathrm{d}t_k \\ = {} & P_\mu^{(1/\gamma)}[k] P_\nu^{(1/\gamma)}[k+\beta/\gamma-1] \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{\Gamma(\alpha+\beta+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma(\alpha+\beta+(2k-i-j)\gamma+\mu_i+\nu_j)} \\ & \times \prod_{i=1}^{k} \frac{\Gamma(\alpha+(k-i)\gamma+\mu_i)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-\ell-i-1)\gamma+\mu_i)\Gamma(1+\gamma)},\end{aligned} \tag{4.3.1}$$

where μ is a partition with $\ell(\mu) \leqslant k$ and ν is a partition such that $\ell(\nu) \leqslant \ell$ for some integer ℓ. The above formula was discovered by Alba *et. al* in [AFLT11, Appendix A]. For $\lambda = \mu = 0$ it reduces to the Selberg integral (4.2.1). If $\mu = 0$ it reduces to the Kadell integral (1.1.4) and for $\beta = \gamma$ the Hua–Kadell integral. It should be noted that the proof provided in [AFLT11, Appendix A] is only valid in the case that α and β are positive integer multiples of γ, and adds the restriction that $\ell(\nu) \leqslant k$. The technique we used to prove the Askey–Habsieger–Kadell integral may in fact be used to prove a q-analogue of (4.3.1) for more general parameters α and β. In doing so we take advantage of the full Cauchy identity for Macdonald polynomials, rather than the specialised Kaneko–Macdonald q-binomial theorem.

Theorem 4.6. *Let μ be a partition with $\ell(\mu) \leqslant k$ and ν a partition with $\ell(\nu) \leqslant \ell$ for another integer ℓ. Then for $\alpha, \beta \in \mathbb{C}$ and γ a positive integer we have*

$$\int\limits_{[0,1]^k} P_\mu(t;q,q^\gamma) P_\nu\biggl(\biggl[q^{\beta-\gamma}t + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\biggr];q,q^\gamma\biggr) \prod_{i=1}^k t_i^{\alpha-1}(qt_i;q)_{\beta-1} \tag{4.3.2}$$
$$\times \prod_{1\leqslant i<j\leqslant k} t_i^{2\gamma}(t_j q^{1-\gamma}/t_i;q)_{2\gamma}\, \mathrm{d}_q t_1 \cdots \mathrm{d}_q t_k$$
$$= q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} P_\lambda([\langle 0\rangle_k];q,q^\gamma) P_\mu\biggl(\biggl[\frac{1-q^{\beta+(k-1)\gamma}}{1-q^\gamma}\biggr];q,q^\gamma\biggr)$$
$$\times \prod_{i=1}^k \frac{\Gamma_q(\alpha+(k-i)\gamma+\mu_i)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(1+i\gamma)}{\Gamma_q(\alpha+\beta+(2k-\ell-i-1)\gamma+\mu_i)\Gamma_q(1+\gamma)}$$
$$\times \prod_{i=1}^k \prod_{j=1}^\ell \frac{\Gamma_q(\alpha+\beta+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma_q(\alpha+\beta+(2k-i-j)\gamma+\mu_i+\nu_j)},$$

for $\mathrm{Re}(\alpha) > -\mu_k$*,* $\beta \neq 0, -1, -2, \ldots$*, and* γ *a positive integer.*

Remark. For $\lambda = \mu = 0$ the integral reduces to the Askey–Habsieger–Kadell integral of Theorem 4.4. For $\mu = 0$ this reduces to the q-analogue of Kadell's integral

$$\int\limits_{[0,1]^k} P_\mu(t;q,q^\gamma) \prod_{i=1}^k t_i^{\alpha-1}(qt_i;q)_{\beta-1} \prod_{1\leqslant i<j\leqslant k} t_i^{2\gamma}(t_j q^{1-\gamma}/t_i;q)_{2\gamma}\, \mathrm{d}_q t$$
$$= q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} P_\mu[\langle 0\rangle_k] \prod_{i=1}^k \frac{\Gamma_q(\alpha+(k-i)\gamma+\mu_i)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(1+i\gamma)}{\Gamma_q(\alpha+\beta+(2k-i-1)\gamma+\mu_i)\Gamma_q(1+\gamma)}.$$

due to Kaneko [Kan96, Proposition 5.2] and Macdonald [Mac15, p. 376]. When $\beta = \gamma$ (4.3.2) becomes a q-analogue of the Hua–Kadell integral due to Warnaar [War05, Theorem 1.4].

Proof of Theorem 4.6. During this proof we suppress all q, t-dependence from Macdonald polynomials for notational convenience. The proof follows essentially the same theme as that of Theorem 4.4. Here we first begin with the plethystically substituted Cauchy identity

$$\sum_\lambda P_\lambda(X) Q_\lambda\biggl[Y + \frac{1-b}{1-t}\biggr] = \prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty} \prod_{x\in X} \frac{(bx;q)_\infty}{(x;q)_\infty}.$$

Note that this is simply (3.4.16) with $a = 1$ and the X and Y alphabets swapped. Now specialising $X \mapsto z\langle\mu\rangle_k$ and $Y \mapsto b\langle\nu\rangle_\ell$ for z a free parameter we obtain

$$\sum_{\lambda} z^{|\lambda|} P_\lambda[\langle\mu\rangle_k] Q_\lambda\Big[b\langle\nu\rangle_\ell + \frac{1-b}{1-t}\Big] = \prod_{i=1}^{k} \frac{(bzq^{\mu_i}t^{k-i};q)_\infty}{(zq^{\mu_i}t^{k-i};q)_\infty} \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(bzq^{\mu_i+\nu_j}t^{k+\ell-i-j+1};q)_\infty}{(bzq^{\mu_i+\nu_j}t^{k+\ell-i-j};q)_\infty}.$$

The next step is to apply the evaluation symmetries of Theorem 3.6 and Corollary 3.7. In order to do so we must first scale $b \mapsto bt^{-\ell}$. Now applying the symmetries yields

$$\sum_{\lambda} \frac{z^{|\lambda|} c_\lambda(q,t)}{c'_\lambda(q,t)} \frac{P_\lambda[\langle 0\rangle_k] P_\lambda[\frac{1-b}{1-t}]}{P_\mu[\langle 0\rangle_k] P_\nu[\frac{1-b}{1-t}]} P_\mu[\langle\lambda\rangle_k] P_\nu\Big[bt^{-k}\langle\lambda\rangle_k + \frac{1-bt^{-k}}{1-t}\Big] = \prod_{i=1}^{k} \frac{(bzq^{\mu_i}t^{k-\ell-i};q)_\infty}{(zq^{\mu_i}t^{k-i};q)_\infty} \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(bzq^{\mu_i+\nu_j}t^{k-i-j+1};q)_\infty}{(bzq^{\mu_i+\nu_j}t^{k-i-j};q)_\infty}.$$

Proposition 3.9 may be applied to evaluate the Macdonald polynomials indexed by λ so that we arrive at the identity

$$\begin{aligned}\sum_{\lambda} &\frac{z^{|\lambda|} t^{2n(\lambda)} (t^k, b; q, t)_\lambda}{c_\lambda(q,t) c'_\lambda(q,t)} P_\mu[\langle\lambda\rangle_k] P_\nu\Big[bt^{-k}\langle\lambda\rangle_k + \frac{1-bt^{-k}}{1-t}\Big] \\ &= P_\mu[\langle 0\rangle_k] P_\nu\Big[\frac{1-b}{1-t}\Big] \prod_{i=1}^{k} \frac{(bzq^{\mu_i}t^{k-\ell-i};q)_\infty}{(zq^{\mu_i}t^{k-i};q)_\infty} \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(bzq^{\mu_i+\nu_j}t^{k-i-j+1};q)_\infty}{(bzq^{\mu_i+\nu_j}t^{k-i-j};q)_\infty}.\end{aligned} \tag{4.3.3}$$

The left-hand side of (4.3.3) is almost the left-hand side of (4.2.6) from the proof of the Askey–Habsieger–Kadell integral, however with the insertion of two Macdonald polynomials. Therefore if we set the parameters so that $(b,t,z) \mapsto (q^{\beta+(k-1)\gamma}, q^\gamma, q^\alpha)$ we may make the same manipulations as before since these do not effect the Macdonald polynomials present in the sum. Hence we have the formula

$$\begin{aligned}\sum_{\lambda} P_\mu[\langle\lambda\rangle_k] P_\nu\Big[q^{\beta-\gamma}\langle\lambda\rangle_k + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\Big] \prod_{i=1}^{k} q^{2(i-1)\lambda_i+\alpha\lambda_i} (q^{1+(k-i)\gamma+\lambda_i};q)_{\beta-1} \\ \times \prod_{1\leqslant i<j\leqslant k} (1-q^{(j-i)\gamma+\lambda_i-\lambda_j})(q^{1+(j-i-1)\gamma+\lambda_i-\lambda_j};q)_{2\gamma-1} \\ = P_\mu[\langle 0\rangle_k] P_\nu\Big[\frac{1-q^{\beta+(k-1)\gamma}}{1-q^\gamma}\Big] \prod_{i=1}^{k} \frac{(q^{1+(k-i)\gamma}, q^{\alpha+\beta+\mu_i+(2k-\ell-i-1)\gamma};q)_\infty}{(q^{\beta+(k-i)\gamma}, q^{\alpha+\mu_i+(k-i)\gamma};q)_\infty} \\ \times \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(q^{\alpha+\beta+\mu_i+\nu_j+(2k-i-j)\gamma};q)_\infty}{(q^{\alpha+\beta+\mu_i+\nu_j+(2k-i-j-1)\gamma};q)_\infty} \prod_{1\leqslant i<j\leqslant k} \frac{(q^{1+(j-i-1)\gamma}, q^{(j-i)\gamma};q)_\infty}{(q^{1+(j-i)\gamma}, q^{(j-i+1)\gamma};q)_\infty}.\end{aligned}$$

We are again in a position to apply Lemma 4.5. The function f_ν in our case is

$$f_\lambda = q^{\alpha|\lambda|} P_\mu(q^\lambda) P_\nu\left[q^{\beta-\gamma}(q^\lambda) + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\right] \prod_{i=1}^{k} (q^{1+\lambda_i};q)_{\beta-1} \times \prod_{1\leqslant i<j\leqslant k} q^{(2\gamma-1)\lambda_j}(q^{\lambda_j}-q^{\lambda_i})(q^{1-\gamma+\lambda_i-\lambda_j};q)_{2\gamma-1}.$$

Here when we write q^λ inside a Macdonald polynomial we mean the alphabet $\{q^{\lambda_1}, q^{\lambda_2}, \dots, q^{\lambda_k}\}$, which may be written plethystically as $q^{\lambda_1}+q^{\lambda_2}+\cdots+q^{\lambda_k}$. The vanishing condition is again satisfied thanks to the product over $i<j$ in the summand. Furthermore the symmetry condition is satisfied as the extra factors are symmetric functions. We note that we have picked up some extra powers of q as in the Askey–Habsieger–Kadell case. Therefore we have

$$\begin{aligned}
&\sum_\sigma q^{\alpha|\sigma|} P_\mu(q^\sigma) P_\nu\left[q^{\beta-\gamma}(q^\sigma) + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\right] \prod_{i=1}^{k} (q^{1+\sigma_i};q)_{\beta-1} \\
&\qquad \times \prod_{1\leqslant i<j\leqslant k} q^{(2\gamma-1)\sigma_j} \frac{(q^{\sigma_j}-q^{\sigma_i})(q^{\sigma_i}-q^{\sigma_j+\gamma})}{q^{\sigma_i}-q^{\sigma_j}} (q^{1-\gamma+\sigma_i-\sigma_j};q)_{2\gamma-1} \\
&= q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} \frac{(q^\gamma;q^\gamma)_k}{(1-q^\gamma)^k} P_\mu[\langle 0\rangle_k] P_\nu\left[\frac{1-q^{\beta+(k-1)\gamma}}{1-q^\gamma}\right] \\
&\quad \times \prod_{i=1}^{k} \frac{(q^{1+(k-i)\gamma}, q^{\alpha+\beta+\mu_i+(k-i-1)\gamma};q)_\infty}{(q^{\beta+(k-i)\gamma}, q^{\alpha+\mu_i+(k-i)\gamma};q)_\infty} \prod_{1\leqslant i<j\leqslant k} \frac{(q^{1+(j-i-1)\gamma}, q^{(j-i)\gamma};q)_\infty}{(q^{1+(j-i)\gamma}, q^{(j-i+1)\gamma};q)_\infty} \\
&\quad \times \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{(q^{\alpha+\beta+\mu_i+\nu_j+(2k-i-j)\gamma};q)_\infty}{(q^{\alpha+\beta+\mu_i+\nu_j+(2k-i-j-1)\gamma};q)_\infty}.
\end{aligned}$$

Multiplying through by $(1-q)^k$ and again writing everything in terms of q-gamma functions we obtain

$$\begin{aligned}
&\int_{[0,1]^k} P_\mu(t) P_\nu\left[q^{\beta-\gamma} t + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\right] \prod_{i=1}^{k} t_i^{\alpha-1}(qt_i);q)_{\beta-1} \\
&\qquad\qquad \times \prod_{1\leqslant i<j\leqslant k} t_i^{2\gamma}(t_j q^{1-\gamma}/t_i;q)_{2\gamma}\, \mathrm{d}_q t \\
&= q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} P_\mu[\langle 0\rangle_k] P_\nu\left[\frac{1-q^{\beta+(k-1)\gamma}}{1-q^\gamma}\right] \\
&\quad \times \prod_{i=1}^{k} \frac{\Gamma_q(\alpha+(k-i)\gamma+\mu_i)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(1+i\gamma)}{\Gamma_q(\alpha+\beta+(2k-\ell-i-1)\gamma+\mu_i)\Gamma_q(1+\gamma)} \\
&\quad \times \prod_{i=1}^{k}\prod_{j=1}^{\ell} \frac{\Gamma_q(\alpha+\beta+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma_q(\alpha+\beta+(2k-i-j)\gamma+\mu_i+\nu_j)}.
\end{aligned}$$

This is as desired. As before the conditions of α and β come from the q-gamma functions occurring on the right-hand side and the absolute convergence requirement. ■

Now taking the limit as $q \to 1^-$ gives the AFLT integral for γ a positive integer. This restriction may be removed by adjusting the limiting procedure to avoid the use of Lemma 4.5. This is the content of the A_n generalisation we obtain in the next chapter.

CHAPTER 5

A_n SELBERG INTEGRALS

In this chapter we discuss the connection between generalised Selberg integrals and simple Lie algebras arising from the Knizhnik–Zamolodchikov (KZ) equations. We first describe how solutions to the KZ equations are related to hypergeometric integrals. This is followed by an explanation of the A_n *case of the Mukhin–Varchenko conjecture due to Warnaar. We conclude by using the Cauchy identities of Chapter 3 to generalise Warnaar's* A_n *integral to the case of Alba et al.*

5.1 $\mathfrak{g}$-Selberg integrals and the Mukhin–Varchenko conjecture

Due to the appearance of the Vandermonde product in the integrand of the Selberg integral, it is natural to associate it with the root system A_{k-1}, where k is the number of variables in the integral. Macdonald's conjectures show that this is a fruitful association. However we wish to associate the Selberg integral to the Lie algebra $\mathfrak{sl}_2 = \mathrm{A}_1$ (we identify $\mathfrak{sl}_{n+1}$ and A_n throughout this section). Such a viewpoint originates from the relationship between Knizhnik–Zamolodchikov equations and hypergeometric integrals.

The Knizhnik–Zamolodchikov equations first arose in the work of Knizhnik and Zamolodchikov in two-dimensional conformal field theory [KZ86]. Although they may be described much more generally, we restrict ourselves to the following case. Let $\mathfrak{g}$ be a simple Lie algebra of rank n with simple roots $\tilde{\alpha}_i$ (we use this notation to avoid confusion with parameters α_i in our Selberg integrals) and Chevally generators e_i, f_i, and h_i for $1 \leqslant i \leqslant n$. Furthermore let V_λ and V_μ be highest weight modules for $\mathfrak{g}$ with highest weights λ and μ respectively. The KZ equations may then be stated as

$$\kappa\frac{\partial u}{\partial z} = \frac{\Omega}{z-w}u \quad \text{and} \quad \kappa\frac{\partial u}{\partial w} = \frac{\Omega}{w-z}u. \tag{5.1.1}$$

Here $u(z,w)$ is a function taking values in the tensor product $V_\lambda \otimes V_\mu$ and Ω is the Casimir element. The connection between (5.1.1) and hypergeometric integrals first appeared in the work of Schechtman and Varchenko [SV91]. They constructed solutions to the KZ equations for an arbitrary simple Lie algebra in terms of $k_1 + \cdots + k_n$ dimensional hypergeometric integrals. In our case their construction may be described as follows. Let $u(z,w)$ be given by

$$u(z,w) = \sum u_{I,J}(z,w) f^I v_\lambda \otimes f^J v_\mu$$

where

$$u_{I,J}(z,w) = \int_\gamma \Psi(z,w;t)\omega_{I,J}(z,w;t)\,\mathrm{d}t.$$

The above sum is over all ordered multisets I, J of $\{1, \dots, n\}$ such that i appears precisely k_i times in $I \cup J$. Also v_λ, v_μ are the highest weight vectors of V_λ and V_μ respectively, $f^I v_\lambda = (\prod_{i \in I} f_i) v_\lambda$, and γ is a suitably chosen integration cycle. The function $\omega_{I,J}(z, w; t)$ is a rational function that we will have no need for. However the function $\Psi(z, w; t)$ is known as the phase function (or master function) and is defined as follows. The first k_i integration variables are attached to $\tilde{\alpha}_i$ so that $\tilde{\alpha}_{t_j} := \tilde{\alpha}_i$ provided $k_1 + \cdots + k_{i-1} < j \leqslant k_1 + \cdots + k_i$. The function $\Psi(z, w; t)$ is then given by

$$\Psi(z, w; t) = (z - w)^{(\lambda,\mu)/\kappa} \prod_{i=1}^{k} (t_i - z)^{-(\lambda, \tilde{\alpha}_{t_i})/\kappa} (t_i - w)^{-(\mu, \tilde{\alpha}_{t_i})/\kappa} \times \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)^{(\tilde{\alpha}_{t_i}, \tilde{\alpha}_{t_j})/\kappa}.$$

Here $(\cdot, \cdot)$ is the standard bilinear form on the dual of the Cartan subalgebra. In order to obtain explicit Selberg-like integrals we work with the normalised phase function

$$\Psi(t) = \prod_{i=1}^{k} t_i^{-(\lambda, \tilde{\alpha}_{t_i})/\kappa} (1 - t_i)^{-(\mu, \tilde{\alpha}_{t_i})/\kappa} \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)^{(\tilde{\alpha}_{t_i}, \tilde{\alpha}_{t_j})/\kappa}. \tag{5.1.2}$$

Now take $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{A}_1$ with the single fundamental weight Λ_1. Then if we write $\lambda = \lambda_1 \Lambda_1$ and $\mu = \mu_1 \Lambda_1$ the function (5.1.2) becomes

$$\Psi(t) = \prod_{i=1}^{k} t_i^{-\lambda_1/\kappa} (1 - t_i)^{-\mu_1/\kappa} \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)^{2/\kappa}.$$

This is simply the integrand of the Selberg integral with $(\alpha, \beta, \gamma) \mapsto (1 - \lambda_1/\kappa, 1 - \mu_2/\kappa, 1/\kappa)$. Indeed the integral associated to this normalised phase function is the Selberg integral. This specific case motivated Mukhin and Varchenko to make the following much more general conjecture.

Conjecture 5.1 ([MV00, Conjecture 1])**.** *If the space of singular vectors of weight $\lambda + \mu - \sum_{i=1}^{n} k_i \tilde{\alpha}_i$ is one-dimensional, then the integral*

$$\int_{\Delta} \Psi(t) \, \mathrm{d}t$$

evaluates as a product of gamma functions. Here $\Delta \subset [0, 1]^k$ is a real domain of integration not specified.

This conjecture is somewhat remarkable. Neither the exact domain of integration or the form the product of gamma functions should take is specified. Currently a satisfactory solution exists only in the case of A_n. The A_2 case being handled by Tarasov and Varchenko [TV03, Theorem 3.2], and the general A_n case by Warnaar [War09, Theorem 1.2]. It should be pointed out that some low-rank cases have been computed in types B,C and D by Mimachi and Takamuchi by iterating the Euler beta integral [MT05]. These cases are not instructive and the technique is not applicable in higher ranks. No other progress currently exists in the literature for these types.

5.2 The case $\mathfrak{g} = \mathrm{A}_n$

We will now describe the A_n Selberg integral due to Warnaar [War09, Theorem 1.2]. This includes a detailed description of the domain of integration, which becomes much more complicated when passing to higher rank cases.

Throughout this section we consider the KZ equations for $\mathfrak{g} = \mathfrak{sl}_{n+1} = \mathrm{A}_n$. Denote by $\Lambda_1, \dots, \Lambda_n$ the fundamental weights so that $(\Lambda_i, \tilde{\alpha}_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta. As mentioned in the previous section the Selberg integral corresponds to the case of $\mathfrak{g} = \mathrm{A}_1$. In general for A_n we write the highest weights of V_λ and V_μ as $\lambda = \sum_{i=1}^n \lambda_i \Lambda_i$ and $\mu = \sum_{i=1}^n \mu_i \Lambda_i$ respectively. We write the $k_1 + \cdots + k_n$ integration variables as

$$t^{(s)} = (t_1^{(s)}, \dots, t_{k_s}^{(s)})$$

and $t = (t^{(1)}, \dots, t^{(n)})$. We also make use of the Vandermonde-type products for alphabets $x = (x_1, \dots, x_\ell)$ and $y = (y_1, \dots, y_m)$, which are given by

$$\Delta(x) = \prod_{1 \leqslant i < j \leqslant \ell} (x_i - x_j), \qquad \Delta(x, y) = \prod_{i=1}^{\ell} \prod_{j=1}^{m} (x_i - y_j).$$

The specialised phase function is then

$$\Psi(t) = \prod_{s=1}^{n} \left(|\Delta(t^{(s)})|^{2/\kappa} \prod_{i=1}^{k_s} (t_i^{(s)})^{-\lambda_s/\kappa} (1 - t_i^{(s)})^{-\mu_s/\kappa} \right) \prod_{s=1}^{n-1} |\Delta(t^{(s)}, t^{(s+1)})|^{-1/\kappa}.$$

In searching for a proof of Conjecture 5.1, Tarasov and Varchenko proved the following integral evaluation.

Theorem 5.2 ([TV03, Theorem 3.2])**.** *Let $0 \leqslant k_1 \leqslant k_2$ be nonnegative integers and define $t = (t_1, \dots, t_{k_1})$ and $s = (s_1, \dots, s_{k_2})$. Consider $\alpha_1, \alpha_2, \beta, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2), \mathrm{Re}(\beta) > 0$ and $|\gamma|$ is sufficiently small. Then*

$$\int_{C_\gamma^{k_1,k_2}[0,1]} \prod_{i=1}^{k_1} t_i^{\alpha_1 - 1} \prod_{i=1}^{k_2} s_i^{\alpha_2 - 1} (1 - s_i)^{\beta - 1} \times |\Delta(t)|^{2\gamma} |\Delta(s)|^{2\gamma} |\Delta(t, s)|^{-\gamma} \, \mathrm{d}t \, \mathrm{d}s \tag{5.2.1}$$

$$= \prod_{1 \leqslant r \leqslant s \leqslant 2} \prod_{i=1}^{k_r - k_{r-1}} \frac{\Gamma(\alpha_r + \cdots + \alpha_s + (r - s + i - 1)\gamma)}{\Gamma(\beta_s + \alpha_r + \cdots + \alpha_s + (k_s - k_{s+1} + i + r - s - 2)\gamma)}$$

$$\times \prod_{i=1}^{k_1} \frac{\Gamma(1 + (i - k_2 - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)} \prod_{j=1}^{k_2} \frac{\Gamma(\beta + (j - 1)\gamma)\Gamma(j\gamma)}{\Gamma(\gamma)}.$$

In the above we define $k_3 := 0$.

Here the integration domain is much more complicated than the domain of the original Selberg integral, however one has

$$C_\gamma^{0,k_2}[0,1] = \{(s_1,\dots,s_{k_2}) \in \mathbb{R}^{k_2} : 0 < s_1 < \dots < s_{k_2} < 1\}.$$

Hence (5.2.1) reduces to the Selberg integral for $k_1 = 0$. To see this note that as the integrand of the Selberg integral (4.2.1) is symmetric we have

$$\int\limits_{C^{0,k_2}[0,1]} = \int\limits_{0<s_1<\dots<s_{k_2}<1} \longrightarrow \frac{1}{k_2!}\int\limits_{[0,1]^{k_2}} .$$

In terms of the KZ equations, the Tarasov–Varchenko integral corresponds to $\mathfrak{g} = \mathfrak{sl}_3 = \mathrm{A}_2$ with highest weights $\lambda = \lambda_1\Lambda_1 + \lambda_2\Lambda_2$ and $\mu = \mu_2\Lambda_2$. We identify the exponents of the Selberg integral with those of the normalised phase function by

$$\lambda_i = (1-\alpha_i)/\kappa, \qquad \mu_i = (1-\beta_i)/\kappa, \qquad \gamma = 1/\kappa$$

so that the normalised phase function agrees with the integrand of (5.2.1). Here this implies that $\beta_1 = 1$ so that the product

$$\prod_{i=1}^{k_1}(1-t_i)^{\beta_1-1}$$

does not occur in the integrand. However there is a symmetry in the parameters. Indeed interchanging the weights λ and μ will interchange the parameters in a way that it is possible to obtain either two α parameters and one free β, or one free α and two β parameters. This is instructive for the general case.

In [War09], Warnaar extends the integral (5.2.1) to $\mathfrak{g} = \mathrm{A}_n$ using the theory of Macdonald polynomials. The evaluation stated there takes highest weights $\lambda = \lambda_n\Lambda_n$ and $\mu = \sum_{i=1}^n \mu_i\Lambda_i$. These weights are chosen so that the space of singular vectors of weight $\lambda + \mu - \sum_{i=1}^n k_i\tilde{\alpha}_i$ is one-dimensional. Here it is necessary to impose that $k_1 \leqslant k_2 \leqslant \dots \leqslant k_n$. This is essentially a consequence of the h-Pieri rule for the classical Schur functions, which states that for ν, ω partitions one has

$$s_\nu h_r = \sum_\omega s_\omega,$$

where the sum is over all $\omega \supset \nu$ such that ω/ν is a horizontal r-strip [Mac15, p. 73]. Let $\tilde{\lambda}$ and $\tilde{\mu}$ be partitions obtained from the weights λ and μ. In this identification $\tilde{\lambda} = \big((\lambda_n)^n\big)$ is a rectangular shape, while $\tilde{\mu}$ with $\ell(\tilde{\mu}) \leqslant n$ has arbitrary shape. After taking complements with respect to a sufficiently large rectangular partition one obtains a decomposition of the tensor product

$V_\lambda \otimes V_\mu$ into a direct sum of highest weight modules with multiplicity one. This is the reason for choosing such highest weight vectors.

Considering λ and μ as above Warnaar claims the following integral formula.

Theorem 5.3 ([War09, Theorem 1.2]). *Let n be a positive integer. Consider nonnegative integers $k_1, \dots, k_n$ such that $0 \leqslant k_1 \leqslant \dots \leqslant k_n$ and define $k_0 = k_{n+1} := 0$. Take $\alpha_1, \dots, \alpha_n, \beta, \gamma \in \mathbb{C}$ such that*

$$\mathrm{Re}(\alpha_1), \dots, \mathrm{Re}(\alpha_n) > 0, \quad \mathrm{Re}(\beta) > 0, \quad -\min\left\{\frac{\mathrm{Re}(\beta)}{k_n - 1}, \frac{1}{k_n}\right\} < \mathrm{Re}(\gamma) < \frac{1}{k_n}$$

and

$$-\frac{\mathrm{Re}(\alpha_r)}{k_r - k_{r-1} - 1} < \mathrm{Re}(\gamma) < \frac{\mathrm{Re}(\alpha_r + \dots + \alpha_s)}{s - r}, \qquad 1 \leqslant r \leqslant s \leqslant n.$$

Then

$$\begin{aligned}\int\limits_{C_\gamma^{k_1,\dots,k_n}[0,1]} & \prod_{r=1}^{n} \left|\Delta(t^{(r)})\right|^{2\gamma} \prod_{i=1}^{k_r} (t_i^{(r)})^{\alpha_r - 1} (1 - t_i^{(s)})^{\beta_s - 1} \prod_{r=1}^{n-1} \left|\Delta(t^{(r)}, t^{(r+1)})\right|^{-\gamma} \mathrm{d}t \\ &= \prod_{1 \leqslant r \leqslant s \leqslant n} \prod_{i=1}^{k_r - k_{r-1}} \frac{\Gamma(\alpha_r + \dots + \alpha_s + (r - s + i - 1)\gamma)}{\Gamma(\beta_s + \alpha_r + \dots + \alpha_s + (k_s - k_{s+1} + i + r - s - 2)\gamma)} \\ &\times \prod_{s=1}^{n} \prod_{i=1}^{k_s} \frac{\Gamma(\beta_s + (i - k_{s+1} - 1)\gamma)\Gamma_q(i\gamma)}{\Gamma(\gamma)}\end{aligned}$$

where we define $\beta_1 = \dots = \beta_{n-1} := 1$ and $\beta_n = \beta$.

Note that whenever a zero denominator occurs in the conditions on the parameters in the above theorem this is to be interpreted as $\pm\infty$ with the same sign as the numerator. This ensures that the conditions may be written succinctly for $k_{s+1} > k_s$.

An important property of the above integral is the following reduction property. Denote the above integral by

$$I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\alpha; \beta_1, \dots, \beta_n; \gamma).$$

For another positive integer m such that $0 \leqslant m \leqslant n$ we have that

$$I^{\mathrm{A}_n}_{\underbrace{0,\dots,0}_{n-m},k_1,\dots,k_m}(\alpha; \beta_1, \dots, \beta_n; \gamma) = I^{\mathrm{A}_m}_{k_1,\dots,k_m}(\alpha; \beta_{n-m+1}, \dots, \beta_n; \gamma).$$

In particular, the chain of integration reduces in the same way.

We choose not to provide an explicit proof of Theorem 5.3, instead noting that it is a specific case of Theorem 5.4 below. However we do wish to describe the chain of integration, denoted $C_\gamma^{k_1,\dots,k_n}[0,1]$, as this depends heavily on the relative ordering between sets of integration variables.

Throughout this description we fix integers $0 \leqslant k_1 \leqslant \cdots \leqslant k_n$ for a positive integer n. In Warnaar's proof of Theorem 5.3 one obtains a q-integral over the domain $D^{k_1,\dots,k_n}[0,1] \subseteq [0,1]^{k_1+\cdots+k_n}$, which is the set of all points

$$(t^{(1)},\dots,t^{(n)}) = (t_1^{(1)},\dots,t_{k_1}^{(1)},\dots,t_1^{(n)},\dots,t_{k_n}^{(n)}) \in [0,1]^{k_1+\cdots+k_n}$$

subject to

$$0 \leqslant t_1^{(s)} \leqslant \cdots \leqslant t_{k_s}^{(s)} \leqslant 1 \tag{5.2.2}$$

for $1 \leqslant s \leqslant n$, and

$$t_i^{(s)} \leqslant t_{i-k_s+k_{s+1}}^{(s+1)} \tag{5.2.3}$$

for $1 \leqslant i \leqslant k_s$ and $1 \leqslant s \leqslant n-1$. We now wish to fix a complete ordering between sets of integration variables $t^{(s)}, t^{(s+1)}$. Such an ordering may be described by a map

$$M_s : \{1,\dots,k_s\} \to \{1,\dots,k_{s+1}\}$$

such that $M_s(i) \leqslant M_s(i+1)$ and $1 \leqslant M_s(i) \leqslant k_{s+1}-k_s+i$, so that

$$t_{M_s(i)-1}^{(s+1)} \leqslant t_i^{(s)} \leqslant t_{M_s(i)}^{(s+1)}. \tag{5.2.4}$$

In the above we define $t_0^{(s+1)} := 0$. We now define the sets

$$D_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n} \subseteq D^{k_1,\dots,k_n}[0,1]$$

by demanding that (5.2.4) holds. Hence we may write $D^{k_1,\dots,k_n}$ as a chain,

$$D^{k_1,\dots,k_n}[0,1] = \sum_{M_1,\dots,M_{n-1}} D_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}[0,1]$$

where the sum is over all admissible maps M_s. Now define the quantity

$$F_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}(\gamma) = \prod_{s=1}^{n-1}\prod_{i=1}^{k_s} \frac{\sin(\pi(i+k_{s+1}-k_s-M_s(i)+1)\gamma)}{\sin(\pi(i+k_{s+1}-k_s)\gamma)}$$

where γ satisfies the conditions of Theorem 5.3. Then the chain $C_\gamma^{k_1,\dots,k_n}[0,1]$ is defined as

$$C_\gamma^{k_1,\dots,k_n}[0,1] = \sum_{M_1,\dots,M_{n-1}} F_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}(\gamma) D_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}[0,1]. \tag{5.2.5}$$

We will see how this chain arises more explicitly during the proof of Theorem 5.4 below.

5.3 An A_n Alba–Fateev–Litvinov–Tarnopolskiy integral

In this section we prove a generalisation of the AFLT integral to A_n. The technique we apply uses the generalised Cauchy identities of Chapter 3. This follows the method for proving Selberg integrals developed by Warnaar [War05, War09, War10]. More specifically the use of the generalised evaluation symmetry for Macdonald polynomials (Corollary 3.7) was introduced in [War10] in order to prove $\mathfrak{sl}_3$ Selberg integrals.

Theorem 5.4 (A_n AFLT integral)**.** *Consider the same conditions as Theorem 5.3. Then for partitons μ and ν such that $\ell(\mu) \leqslant k$ and $\ell(\nu) \leqslant \ell$ for some nonnegative integer ℓ we have*

$$\int\limits_{C_\gamma^{k_1,\dots,k_n}[0,1]} P_\mu^{(1/\gamma)}(t^{(1)}) P_\nu^{(1/\gamma)}[t^{(n)} + \beta/\gamma - 1] \prod_{i=1}^{k_n} (1 - t_i^{(n)})^{\beta-1} \tag{5.3.1}$$

$$\times \prod_{r=1}^{n}\prod_{i=1}^{k_r} (t_i^{(r)})^{\alpha_r - 1} \big|\Delta(t^{(r)})\big|^{2\gamma} \prod_{r=1}^{n-1} \big|\Delta(t^{(r)}, t^{(r+1)})\big|^{-\gamma}\, \mathrm{d}t^{(1)} \cdots \mathrm{d}t^{(n)}$$

$$= P_\mu^{(1/\gamma)}[k_1] P_\nu^{(1/\gamma)}[k_n + \beta/\gamma - 1] \prod_{s=1}^{n}\prod_{i=1}^{k_s} \frac{\Gamma(\beta_s + (i - k_{s+1} - 1)\gamma)\Gamma_q(i\gamma)}{\Gamma(\gamma)}$$

$$\times \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{\Gamma(\alpha_{r+1} + \cdots + \alpha_s + (r - s + i)\gamma)}{\Gamma(1 + \alpha_{r+1} + \cdots + \alpha_s + (k_s - k_{s+1} + i + r - s - 1)\gamma)}$$

$$\times \prod_{r=1}^{n-1} \Bigg[\prod_{i=1}^{k_{r+1}-k_r} \frac{\Gamma(\alpha_{r+1} + \cdots + \alpha_n + (r - n + i)\gamma)}{\Gamma(\alpha_{r+1} + \cdots + \alpha_n + \beta + (k_n - \ell + r - n + i - 1)\gamma)}$$

$$\times \prod_{i=1}^{k_1} \frac{\Gamma(\alpha_1 + \cdots + \alpha_r + (k_1 - r - i + 1)\gamma + \mu_i)}{\Gamma(1 + \alpha_1 + \cdots + \alpha_r + (k_r - k_{r+1} + k_1 - r - i)\gamma + \mu_i)}$$

$$\times \prod_{j=1}^{\ell} \frac{\Gamma(\alpha_{r+1} + \cdots + \alpha_n + \beta + (k_n + r - n - j)\gamma + \nu_j)}{\Gamma(\alpha_{r+1} + \cdots + \alpha_n + \beta + (k_n + k_{r+1} - k_r + r - n - j)\gamma + \nu_j)} \Bigg]$$

$$\times \prod_{i=1}^{k_1}\prod_{j=1}^{\ell} \frac{\Gamma(\alpha_1 + \cdots + \alpha_n + \beta + (k_n + k_1 - n - i - j)\gamma + \mu_i + \nu_j)}{\Gamma(\alpha_1 + \cdots + \alpha_n + \beta + (k_n + k_1 - n - i - j + 1)\gamma + \mu_i + \nu_j)}$$

$$\times \prod_{i=1}^{k_1} \frac{\Gamma(\alpha_1 + \cdots + \alpha_n + (k_1 - n - i + 1)\gamma + \mu_i)}{\Gamma(\alpha_1 + \cdots + \alpha_n + \beta + (k_n + k_1 - \ell - n - i)\gamma + \mu_i)},$$

where again we define $\beta_1 = \cdots = \beta_{n-1} := 1$, $\beta_n := \beta$, and $k_0 = k_{n+1} := 0$.

Remark. A number of special cases are immediate.

1. For $n=1$ the integral collapses to the formula

$$\int\limits_{C_\gamma^k[0,1]} P_\mu^{(1/\gamma)}(t)P_\nu^{(1/\gamma)}[t+\beta/\gamma-1]|\Delta(t)|^{2\gamma}\prod_{i=1}^k t_i^{\alpha-1}(1-t_i)^{\beta-1}\,\mathrm{d}t$$
$$= P_\mu^{(1/\gamma)}[k]P_\nu^{(1/\gamma)}[k+\beta/\gamma-1]$$
$$\times\prod_{i=1}^k \frac{\Gamma(\beta+(i-1)\gamma)\Gamma(\alpha+(k-i)\gamma+\mu_i)\Gamma(i\gamma)}{\Gamma(\alpha+\beta+(2k-\ell-i-1)\gamma+\mu_i)\Gamma(\gamma)}$$
$$\times\prod_{i=1}^k\prod_{j=1}^{\ell}\frac{\Gamma(\alpha+\beta+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma(\alpha+\beta+(2k-i-j)\gamma+\mu_i+\nu_j)},$$

   where we have replaced $(k_1,\alpha_1,\beta)\mapsto(k,\alpha,\beta)$. This is the AFLT integral (1.1.5). To see this note that we may replace the domain $C_\gamma^k[0,1]$ by $[0,1]^k$ provided we divide through by $k!$. This may be absorbed into the product of gamma functions on the right, and hence the two integral formulas are equivalent.

2. When $\mu=\nu=0$ the integral reduces to the A_n integral of Theorem 5.3.

3. Setting $\nu=0$ gives the A_n analogue of Kadell's integral (1.1.4) due to Warnaar [War09, Theorem 6.1].

4. Unlike Theorem 5.3 the A_n AFLT integral does not satisfy a nice reduction formula. Indeed in order to reduce the rank we must set $k_1=0$, and so we necessarily have $\mu=0$ in this case. Essentially this gives the AFLT integral for rank $n-1$ with $\mu=0$ (up to a trivial notational change). This indicates that the integral of this theorem is not the most general Kadell-type integral for A_n. Due to the reduction behaviour a more general formula would necessarily involve the insertion of more Jack polynomials in the integrand. This is discussed in Section 5.4.

Proof of Theorem 5.4. Recall the notations of Theorem 3.16. This claims that for $0\leqslant k_1\leqslant\cdots\leqslant k_n$ nonnegative integers with $a_r=t^{k_r-k_{r+1}}$ for $1\leqslant r\leqslant n-1$ then

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}}\prod_{r=1}^n P_{\lambda^{(r)}}(X^{(r)})Q_{\lambda^{(r)}}(Y^{(r)})\prod_{r=1}^{n-1}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}}\frac{(a_rqt^{j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(a_rqt^{j-i})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}$$
$$=\prod_{s=1}^{n-1}\prod_{x\in X}\frac{(a_sqz_1\cdots z_sx)_\infty}{(tz_1\cdots z_sx)_\infty}\prod_{1\leqslant r<s\leqslant n-1}\prod_{i=1}^{k_{r+1}-k_r}\frac{(a_sqt^{i-1}z_{r+1}\cdots z_s)_\infty}{(t^iz_{r+1}\cdots z_s)_\infty}$$
$$\times\prod_{r=1}^{n-1}\prod_{y\in Y}\frac{(z_{r+1}\cdots z_{n-1}y/a_r)_\infty}{(z_{r+1}\cdots z_{n-1}y)_\infty}\prod_{x\in X}\prod_{y\in Y}\frac{(tz_1\cdots z_{n-1}xy)_\infty}{(z_1\cdots z_{n-1}xy)_\infty},$$

holds with

$$X^{(1)} := \{x_1, \ldots, x_{k_1}\}, \qquad X^{(r+1)} := \frac{t^{-k_r} - a_r^{-1}}{1-t}, \qquad 1 \leqslant r \leqslant n-1$$

$$Y^{(n)} := \{y_1, y_2, \ldots\}, \qquad Y^{(r)} := z_r \frac{t - a_r q t^{k_{r+1}}}{1-t}, \quad 1 \leqslant r \leqslant n-1.$$

This is an identity in the ring of symmetric functions on the alphabet Y, and so we may make the plethystic substitution $Y \mapsto Y + (1-b)/(1-t)$ for a parameter b. Doing so yields

$$\begin{aligned}
&\sum_{\lambda^{(1)},\ldots,\lambda^{(n)}} P_{\lambda^{(r)}}(X^{(r)}) Q_{\lambda^{(r)}}\Big[Y^{(r)} + \frac{1-b}{1-t}\Big] \prod_{r=1}^{n-1} P_{\lambda^{(r)}}(X^{(r)}) Q_{\lambda^{(r)}}(Y^{(r)}) \\
&\qquad \times \prod_{r=1}^{n-1} \prod_{i=1}^{k_r} \prod_{j=1}^{k_{r+1}} \frac{(a_r q t^{j-i-1})_{\lambda_i^{(r)} - \lambda_j^{(r+1)}}}{(a_r q t^{j-i})_{\lambda_i^{(r)} - \lambda_j^{(r+1)}}} \\
&= \prod_{s=1}^{n-1} \prod_{x \in X} \frac{(a_s q z_1 \cdots z_s x)_\infty}{(t z_1 \cdots z_s x)_\infty} \prod_{1 \leqslant r < s \leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1} \cdots z_s)_\infty}{(t^i z_{r+1} \cdots z_s)_\infty} \\
&\quad \times \prod_{r=1}^{n-1} \Bigg[\prod_{i=1}^{k_{r+1}-k_r} \frac{(b z_{r+1} \cdots z_{n-1} t^{i-1})_\infty}{(z_{r+1} \cdots z_{n-1} t^{i-1})_\infty} \prod_{y \in Y} \frac{(z_{r+1} \cdots z_{n-1} y / a_r)_\infty}{(z_{r+1} \cdots z_{n-1} y)_\infty} \Bigg] \\
&\quad \times \prod_{x \in X} \prod_{y \in Y} \frac{(t z_1 \cdots z_{n-1} x y)_\infty}{(z_1 \cdots z_{n-1} x y)_\infty} \prod_{x \in X} \frac{(b z_1 \cdots z_{n-1} x)_\infty}{(z_1 \cdots z_{n-1} x)_\infty},
\end{aligned}$$

We now introduce partitions μ and ν. We have that $\ell(\mu) \leqslant k_1$ as the alphabet $X^{(1)}$ is finite of cardinality k_1. However the alphabet $Y^{(r)}$ is countably infinite and occurs in the factor $Q_{\lambda^{(n)}}[Y^{(n)} + \frac{1-b}{1-t}]$. Hence the length of ν is not restricted by k_n. Therefore we introduce a positive integer ℓ such that $\ell(\nu) \leqslant \ell$. We make the substitutions $X^{(1)} \mapsto \langle \mu \rangle_{k_1}$ and $Y^{(r)} \mapsto b z_n \langle \nu \rangle_\ell + z_n \frac{1-b}{1-t}$ to give

$$\begin{aligned}
&\sum_{\lambda^{(1)},\ldots,\lambda^{(n)}} z_n^{|\lambda^{(n)}|} P_{\lambda^{(1)}}(\langle \mu \rangle_{k_1}) Q_{\lambda^{(1)}}(Y^{(1)}) P_{\lambda^{(r)}}(X^{(r)}) Q_{\lambda^{(r)}}\Big[b \langle \nu \rangle_\ell + \frac{1-b}{1-t}\Big] \\
&\qquad \times \prod_{r=2}^{n-1} P_{\lambda^{(r)}}(X^{(r)}) Q_{\lambda^{(r)}}(Y^{(r)}) \prod_{r=1}^{n-1} \prod_{i=1}^{k_r} \prod_{j=1}^{k_{r+1}} \frac{(a_r q t^{j-i-1})_{\lambda_i^{(r)} - \lambda_j^{(r+1)}}}{(a_r q t^{j-i})_{\lambda_i^{(r)} - \lambda_j^{(r+1)}}} \\
&= \prod_{s=1}^{n-1} \prod_{i=1}^{k_1} \frac{(a_s z_1 \cdots z_s q^{\mu_i + 1} t^{k_1 - i})_\infty}{(z_1 \cdots z_s q^{\mu_i} t^{k_1 - i + 1})_\infty} \prod_{1 \leqslant r < s \leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1} \cdots z_s)_\infty}{(t^i z_{r+1} \cdots z_s)_\infty} \\
&\quad \times \prod_{r=1}^{n-1} \Bigg[\prod_{i=1}^{k_{r+1}-k_r} \frac{(b z_{r+1} \cdots z_n t^{i-1})_\infty}{(z_{r+1} \cdots z_n t^{i-1})_\infty} \prod_{j=1}^{\ell} \frac{(b z_{r+1} \cdots z_n q^{\nu_j} t^{\ell-j} / a_r)_\infty}{(b z_{r+1} \cdots z_n q^{\nu_j} t^{\ell - j})_\infty} \Bigg] \\
&\quad \times \prod_{i=1}^{k_1} \prod_{j=1}^{\ell} \frac{(b z_1 \cdots z_n q^{\mu_i + \nu_j} t^{k_1 + \ell - i - j + 1})_\infty}{(b z_1 \cdots z_n q^{\mu_i + \mu_j} t^{k_1 + \ell - i - j})_\infty} \prod_{i=1}^{k_1} \frac{(b z_1 \cdots z_n q^{\mu_i} t^{k_1 - i})_\infty}{(z_1 \cdots z_n q^{\mu_i} t^{k_1 - i})_\infty}.
\end{aligned}$$

We wish to use the generalised evaluation symmetry (Corollary 3.7) to interchange the summation variables and the partitions μ and ν. In order to do so we scale $b \mapsto bt^{-\ell}$ and then apply the symmetry, giving

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} z_n^{|\lambda^{(n)}|} P_{\lambda^{(1)}}(\langle 0\rangle_{k_1}) Q_{\lambda^{(n)}}\Big[\frac{1-b}{1-t}\Big] Q_{\lambda^{(1)}}(Y^{(1)}) P_{\lambda^{(n)}}(X^{(r)}) \tag{5.3.2}$$
$$\times P_\mu(\langle\lambda^{(1)}\rangle_{k_1}) P_\nu\Big[bt^{-k_n}\langle\lambda^{(n)}\rangle_{k_n} + \frac{1-bt^{-k_n}}{1-t}\Big]$$
$$\times \prod_{r=2}^{n-1} P_{\lambda^{(r)}}(X^{(r)}) Q_{\lambda^{(r)}}(Y^{(r)}) \prod_{r=1}^{n-1}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \frac{(a_r q t^{j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(a_r q t^{j-i})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}$$
$$= P_\mu(\langle 0\rangle_{k_1}) P_\nu\Big[\frac{1-b}{1-t}\Big] \prod_{s=1}^{n-1}\prod_{i=1}^{k_1} \frac{(a_s z_1\cdots z_s q^{\mu_i+1} t^{k_1-i})_\infty}{(z_1\cdots z_s q^{\mu_i} t^{k_1-i+1})_\infty}$$
$$\times \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1}\cdots z_s)_\infty}{(t^i z_{r+1}\cdots z_s)_\infty}$$
$$\times \prod_{r=1}^{n-1}\Bigg[\prod_{i=1}^{k_{r+1}-k_r} \frac{(bz_{r+1}\cdots z_n t^{i-\ell-1})_\infty}{(z_{r+1}\cdots z_n t^{i-1})_\infty} \prod_{j=1}^{\ell} \frac{(bz_{r+1}\cdots z_n q^{\nu_j} t^{-j}/a_r)_\infty}{(bz_{r+1}\cdots z_n q^{\nu_j} t^{-j})_\infty}\Bigg]$$
$$\times \prod_{i=1}^{k_1}\prod_{j=1}^{\ell} \frac{(bz_1\cdots z_n q^{\mu_i+\nu_j} t^{k_1-i-j+1})_\infty}{(bz_1\cdots z_n q^{\mu_i+\mu_j} t^{k_1-i-j})_\infty} \prod_{i=1}^{k_1} \frac{(bz_1\cdots z_n q^{\mu_i} t^{k_1-\ell-i})_\infty}{(z_1\cdots z_n q^{\mu_i} t^{k_1-i})_\infty}.$$

We now focus on the left-hand side of this identity, denoted LHS(5.3.2). Define $k_0 := 0$ and $b_1,\dots,b_n$ by

$$b_r := qt^{k_r-1}, \quad 1 \leqslant r \leqslant n-1, \quad b_n := b. \tag{5.3.3}$$

Using the specialisation formula from Proposition 3.9 and recalling that $a_r := t^{k_r-k_{r+1}}$ for $1\leqslant r\leqslant n-1$ we may write

$$\mathrm{LHS}(5.3.2) = \sum_{\lambda^{(1)},\dots,\lambda^{(n)}} P_\mu(\langle\lambda^{(1)}\rangle_{k_1}) P_\nu\Big[b_n t^{-k_n}\langle\lambda^{(n)}\rangle_{k_n} + \frac{1-b_n t^{-k_n}}{1-t}\Big]$$
$$\times t^{-|\lambda^{(n)}|} \prod_{r=1}^{n} t^{2n(\lambda^{(r)})+(1-k_{r-1})|\lambda^{(r)}|} z_r^{|\lambda^{(r)}|} \frac{(t^{k_r}, b_r; q,t)_{\lambda^{(r)}}}{c_{\lambda^{(r)}}(q,t) c'_{\lambda^{(r)}}(q,t)}$$
$$\times \prod_{r=1}^{n-1}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \frac{(qt^{k_r-k_{r+1}+j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(qt^{k_r-k_{r+1}+j-i})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}.$$

This may be expressed in terms of shifted factorials, so that

$$\begin{aligned}
\mathrm{LHS}(5.3.2) = \sum_{\lambda^{(1)},\dots,\lambda^{(n)}} & P_\mu(\langle\lambda^{(1)}\rangle_{k_1}) P_\nu\Big[b_n t^{-k_n}\langle\lambda^{(n)}\rangle_{k_n} + \frac{1-b_n t^{-k_n}}{1-t}\Big] \\
& \times t^{2n(\lambda^{(n)})-k_{n-1}|\lambda^{(n)}|} z_n^{|\lambda^{(n)}|} \prod_{i=1}^{k_n} \frac{(b_n t^{1-i})_{\lambda_i^{(n)}}}{(qt^{k_n-i})_{\lambda_i^{(n)}}} \\
& \times \prod_{r=1}^{n-1} t^{2n(\lambda^{(r)})+(1-k_{r-1})|\lambda^{(r)}|} z_r^{|\lambda^{(r)}|} \prod_{1\leqslant i<j\leqslant k_r} \frac{(t^{j-i+1}, qt^{j-i})_{\lambda_i^{(r)}-\lambda_j^{(r)}}}{(t^{j-i}, qt^{j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r)}}} \\
& \times \prod_{r=1}^{n-1}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \frac{(qt^{k_r-k_{r+1}+j-i-1})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}{(qt^{k_r-k_{r+1}+j-i})_{\lambda_i^{(r)}-\lambda_j^{(r+1)}}}.
\end{aligned}$$

Moving terms independent of the summation variables to the right-hand side and rewriting in terms of infinite q-shifted factorials we have the full identity

$$\begin{aligned}
\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} & P_\mu(\langle\lambda^{(1)}\rangle_{k_1}) P_\nu\Big[b_n t^{-k_n}\langle\lambda^{(n)}\rangle_{k_n} + \frac{1-b_n t^{-k_n}}{1-t}\Big] \qquad (5.3.4)\\
& \times \Bigg[t^{-|\lambda^{(n)}|} \prod_{r=1}^{n} t^{2n(\lambda^{(r)})+(1-k_{r-1})|\lambda^{(r)}|} z_r^{|\lambda^{(r)}|} \prod_{i=1}^{k_r} \frac{(q^{1+\lambda_i^{(r)}} t^{k_r-i})_\infty}{(b_r q^{\lambda_i^{(r)}} t^{1-i})_\infty} \\
& \qquad \times \prod_{1\leqslant i<j\leqslant k_r} \frac{(q^{\lambda_i^{(r)}-\lambda_j^{(r)}} t^{j-i}, q^{1+\lambda_i^{(r)}-\lambda_j^{(r)}} t^{j-i-1})_\infty}{(q^{\lambda_i^{(r)}-\lambda_j^{(r)}} t^{j-i+1}, q^{1+\lambda_i^{(r)}-\lambda_j^{(r)}} t^{j-i},)_\infty}\Bigg] \\
& \times \prod_{r=1}^{n-1}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \frac{(q^{\lambda_i^{(r)}-\lambda_j^{(r+1)}} t^{k_r-k_{r+1}+j-i})_\infty}{(q^{\lambda_i^{(r)}-\lambda_j^{(r+1)}} t^{k_r-k_{r+1}+j-i-1})_\infty} \\
= P_\mu(\langle 0\rangle_{k_1}) & P_\nu\Big[\frac{1-b}{1-t}\Big] \prod_{s=1}^{n-1}\prod_{i=1}^{k_1} \frac{(a_s z_1\cdots z_s q^{\mu_i+1} t^{k_1-i})_\infty}{(z_1\cdots z_s q^{\mu_i} t^{k_1-i+1})_\infty} \\
& \times \prod_{r=1}^{n}\Bigg[\prod_{i=1}^{k_r} \frac{(qt^{k_r-i})_\infty}{(b_r t^{1-i})_\infty} \prod_{1\leqslant i<j\leqslant k_r} \frac{(t^{j-i}, qt^{j-i-1})_\infty}{(t^{j-i+1}, qt^{j-i})_\infty}\Bigg] \\
& \times \prod_{r=1}^{n-1}\prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}} \frac{(qt^{k_r-k_{r+1}+j-i})_\infty}{(qt^{k_r-k_{r+1}+j-i-1})_\infty} \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1}\cdots z_s)_\infty}{(t^i z_{r+1}\cdots z_s)_\infty} \\
& \times \prod_{r=1}^{n-1}\Bigg[\prod_{i=1}^{k_{r+1}-k_r} \frac{(bz_{r+1}\cdots z_n t^{i-\ell-1})_\infty}{(z_{r+1}\cdots z_n t^{i-1})_\infty} \prod_{j=1}^{\ell} \frac{(bz_{r+1}\cdots z_n q^{\nu_j} t^{-j}/a_r)_\infty}{(bz_{r+1}\cdots z_n q^{\nu_j} t^{-j})_\infty}\Bigg] \\
& \times \prod_{i=1}^{k_1}\prod_{j=1}^{\ell} \frac{(bz_1\cdots z_n q^{\mu_i+\nu_j} t^{k_1-i-j+1})_\infty}{(bz_1\cdots z_n q^{\mu_i+\mu_j} t^{k_1-i-j})_\infty} \prod_{i=1}^{k_1} \frac{(bz_1\cdots z_n q^{\mu_i} t^{k_1-\ell-i})_\infty}{(z_1\cdots z_n q^{\mu_i} t^{k_1-i})_\infty}.
\end{aligned}$$

We now make the substitutions

$$(b_n, t, z_s) \mapsto (q^{\beta+(k_n-1)\gamma}, q^\gamma, q^{\alpha_s-\gamma}), \quad 1\leqslant s\leqslant n-1, \quad z_n \mapsto q^{\alpha_n} \qquad (5.3.5)$$

into (5.3.4). For variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ define the q-Vandermonde-type products

$$\Delta_\gamma(x;q) = \prod_{1\leqslant i<j\leqslant n} x_j^{2\gamma}\big(1 - q^{(j-i)\gamma}x_i/x_j\big)\big(q^{1+(j-i-1)\gamma}x_i/x_j;q\big)_{2\gamma-1}$$

and

$$\Delta_\gamma(x,y;q) = \prod_{i=1}^{n}\prod_{j=1}^{m} y_j^{-\gamma}\big(q^{1-(i-j-n-m)\gamma}x_i/y_j;q\big)_{-\gamma}.$$

In order to obtain a more explicit q-integral we set $t_i^{(s)} := q^{\lambda_i^{(s)}}$ and $t^{(s)} = (t_1^{(s)}, \dots, t_{k_s}^{(s)})$ (these will become our integration variables). Furthermore define the modified set of variables $\tilde{t}^{(s)} = (t_1^{(s)}q^{(k_s-1)\gamma}, \dots, t_{k_s}^{(s)}q^0)$. We now have the much simpler-looking left-hand side

$$\begin{aligned}\mathrm{LHS}(5.3.4) = \sum_{\lambda^{(1)},\dots,\lambda^{(n)}} & P_\mu(\tilde{t}^{(1)})P_\nu\Big[q^{\beta-\gamma}\tilde{t}^{(n)} + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\Big]\prod_{i=1}^{k_n}(q^{1+(k_n-i)\gamma}t_i^{(n)})_{\beta-1} \\ & \times \prod_{r=1}^{n}\prod_{i=1}^{k_r}(t_i^{(r)})^{\alpha_r}\Delta_\gamma(t^{(r)};q)\prod_{r=1}^{n-1}\Delta_\gamma(t^{(r)},t^{(r+1)};q).\end{aligned}$$

This is the integrand of a (restricted) q-integral. All that now remains is for us to take the limit $q \to 1^-$ in order to obtain a valid integral formula. The restricted q-integral we currently have is over the domain $D^{k_1,\dots,k_n}[0,1]$ as defined in Section 5.2. If we impose the additional assumptions that

$$t_i^{(s)} < t_j^{(s)} \tag{5.3.6a}$$

for $1 \leqslant i < j \leqslant k_s$ and

$$t_i^{(s)} < t_j^{(s+1)} \tag{5.3.6b}$$

for $1 \leqslant i \leqslant k_s$ and $1 \leqslant j \leqslant k_{s+1}$, then we are free to take the limits of the Vandermonde-type products to obtain

$$\begin{aligned}\lim_{q\to1^-}\Delta_\gamma(t^{(s)};q) &= \Delta(-t^{(s)})^{2\gamma}, \\ \lim_{q\to1^-}\Delta_\gamma(t^{(s)},t^{(s+1)};q) &= \Delta(-t^{(s)},-t^{(s+1)})^{-\gamma}.\end{aligned}$$

Recalling the Jack polynomials from Definition 3.5 the limit of the integrand as $q \to 1^-$ is then

$$\begin{aligned}& P_\mu^{(1/\gamma)}[k_1]P_\nu^{(1/\gamma)}[k_n + \beta/\gamma - 1]\prod_{i=1}^{k_n}(1-t_i^{(n)})^{\beta-1} \\ & \qquad \times \prod_{s=1}^{n}\Big(\Delta(-t^{(s)})^{2\gamma}\prod_{i=}^{k_s}(t_i^{(s)})^{\alpha_s-1}\Big)\prod_{s=1}^{n-1}\Delta(-t^{(s)},-t^{(s+1)})^{-\gamma}.\end{aligned} \tag{5.3.7}$$

Unfortunately this is wishful thinking, and the integration variables do not necessarily satisfy such a total ordering as in (5.3.6b). Hence we must consider the limit in the case that $t_i^{(s)} > t_j^{(s+1)}$. Observe that if $t_i^{(s)} < t_j^{(s+1)}$ we have

$$\lim_{q\to 1^-} \left(q^{1-(i-j-k_s-k_{s+1})\gamma} t_i^{(s)}/t_j^{(s+1)};q\right)_{-\gamma} = \left(1 - t_i^{(s)}/t_j^{(s+1)}\right)^{-\gamma}. \tag{5.3.8}$$

Here the t variables occurring on the right-hand side represent integration variables, rather than powers of q. The limit in (5.3.8) is invalid if $t_i^{(s)} > t_j^{(s+1)}$ as the ratio of t variables becomes too large. In order to resolve this we make use of the q-reflection formula [War09, p. 294]

$$\Gamma_q(z)\Gamma_q(1-z) = \frac{2\sqrt{-1}q^{z/2}\theta_1(\sqrt{-1}\,\log q^{z/2};q^{1/2})}{(1-q)\theta_1'(0;q^{1/2})}, \tag{5.3.9}$$

where θ_1 is a theta function as defined in [AAR99, §10.7]. Now define the quantity

$$R_{ij}^{(s)}(\gamma) = \frac{\sin(\pi(i-j-k_s-k_{s+1})\gamma)}{\sin(\pi(i-j-k_s-k_{s+1}+1)\gamma)}. \tag{5.3.10}$$

Following the calculations of [War09, p. 294–295] we see that in the case $t_i^{(s)} > t_j^{(s+1)}$ the limit is

$$\begin{aligned}\lim_{q\to 1^-} \left(q^{1-(i-j-k_s-k_{s+1})\gamma} t_i^{(s)}/t_j^{(s+1)};q\right)_{-\gamma} &= \left(t_i^{(s)}/t_j^{(s+1)} - 1\right)^{-\gamma} R_{ij}^{(s)}(\gamma) \\ &= \left|1 - t_i^{(s)}/t_j^{(s+1)}\right|^{-\gamma} R_{ij}^{(s)}(\gamma).\end{aligned}$$

Therefore we may conclude that in the $q \to 1^-$ limit the integrand is exactly (5.3.7) with $\Delta(-t^{(s)},-t^{(s+1)})^{-\gamma}$ replaced by

$$|\Delta(-t^{(s)},-t^{(s+1)})^{-\gamma}| = |\Delta(t^{(s)},t^{(s+1)})^{-\gamma}|$$

multiplied by a factor of $R_{ij}^{(s)}(\gamma)$ for each occurrence of $t_i^{(s)} > t_j^{(s+1)}$.

With this established we need only determine the chain of integration. This is precisely the chain $C_\gamma^{k_1,\dots,k_n}[0,1]$ of Section 5.2. To see this note that

$$D^{k_1,\dots,k_n}[0,1] = \sum_{M_1,\dots,M_{n-1}} D^{k_1,\dots,k_n}_{M_1,\dots,M_{n-1}}[0,1].$$

From this and the conditions (5.2.4) we pick up a factor of $R_{ij}^{(s)}(\gamma)$ for $1 \leqslant j \leqslant M_s(i) - 1$ as in this case $t_j^{(s+1)} \leqslant t_i^{(s)}$. The product over these factors gives

$$\prod_{i=1}^{M_s(i)-1} R_{ij}^{(s)}(\gamma) = \frac{\sin(\pi(i+k_{s+1}-k_s-M_s(i)+1)\gamma)}{\sin(\pi(i+k_{s+1}-k_s)\gamma)}.$$

Hence in computing the full chain we obtain the decomposition (5.2.5) when sending $q \to 1^-$.

We now multiply both sides of (5.3.4) through by $(1-q)^{k_1+\cdots+k_n}$. Writing the right-hand side in terms of gamma functions yields

$$
\begin{aligned}
&\mathrm{RHS}(5.3.4)\\
&= P_\mu(\langle 0\rangle_{k_1})P_\nu\left[\frac{1-q^{\beta-(k_n-1)\gamma}}{1-q^\gamma}\right]\prod_{s=1}^{n}\prod_{i=1}^{k_s}\frac{\Gamma_q(\beta_s+(i-k_{s+1}-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\gamma)}\\
&\quad\prod_{1\leqslant r<s\leqslant n-1}\prod_{i=1}^{k_{r+1}-k_r}\frac{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_s+(r-s+i)\gamma)}{\Gamma_q(1+\alpha_{r+1}+\cdots+\alpha_s+(k_s-k_{s+1}+i+r-s-1)\gamma)}\\
&\quad\times\prod_{r=1}^{n-1}\Bigg[\prod_{i=1}^{k_{r+1}-k_r}\frac{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+(r-n+i)\gamma)}{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n-\ell+r-n+i-1)\gamma)}\\
&\qquad\times\prod_{i=1}^{k_1}\frac{\Gamma_q(\alpha_1+\cdots+\alpha_r+(k_1-r-i+1)\gamma+\mu_i)}{\Gamma_q(1+\alpha_1+\cdots+\alpha_r+(k_r-k_{r+1}+k_1-r-i)\gamma+\mu_i)}\\
&\qquad\times\prod_{j=1}^{\ell}\frac{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n+r-n-j)\gamma+\nu_j)}{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n+k_{r+1}-k_r+r-n-j)\gamma+\nu_j)}\Bigg]\\
&\quad\times\prod_{i=1}^{k_1}\prod_{j=1}^{\ell}\frac{\Gamma_q(\alpha_1+\cdots+\alpha_n+\beta+(k_n+k_1-n-i-j)\gamma+\mu_i+\nu_j)}{\Gamma_q(\alpha_1+\cdots+\alpha_n+\beta+(k_n+k_1-n-i-j+1)\gamma+\mu_i+\nu_j)}\\
&\quad\times\prod_{i=1}^{k_1}\frac{\Gamma_q(\alpha_1+\cdots+\alpha_n+(k_1-n-i+1)\gamma+\mu_i)}{\Gamma_q(\alpha_1+\cdots+\alpha_n+\beta+(k_n+k_1-\ell-n-i)\gamma+\mu_i)}.
\end{aligned}
$$

With this established we are free to take the $q\to 1^-$ limit, and in doing so we obtain (5.3.1). ■

5.4 Future Directions

The appearance of generalised Selberg integrals with inserted Jack polynomials not only appear in the work of Alba *et al.* [AFLT11], but also in the work of Matsuo and Zhang [MZ11]. Here the authors search for more general Selberg integrals hinted in the remarks following Theorem 5.4. Denote the A_n Selberg integral evaluation of Theorem 5.3 by

$$I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\alpha_1,\dots,\alpha_n;\beta;\gamma).$$

Furthermore denote the Selberg integral with the inclusion of a function $f(t;\beta,\gamma)$ in the integrand by

$$I^{\mathrm{A}_n}_{k_1,\dots,k_n}(f;\alpha_1,\dots,\alpha_n;\beta;\gamma)$$

In their studies of conformal field theory and the Adlay–Gaiotto–Tachikawa (AGT) conjecture, Matsuo and Zhang were interested in computing the *Selberg* *average* for f, which we denote by

$$\langle f\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta;\gamma}=\frac{I^{\mathrm{A}_n}_{k_1,\dots,k_n}(f;\alpha_1,\dots,\alpha_n;\beta;\gamma)}{I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\alpha_1,\dots,\alpha_n;\beta;\gamma)}.$$

The AFLT integral of Theorem 5.4 is then

$$\langle P_\mu(t)P_\nu[t+\beta/\gamma-1]\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta;\gamma}$$

In [MZ11, §3.3] the authors conjecture a Selberg average with the insertion of $n+1$ Schur functions. This is the $\gamma=1$ case of a more general conjecture for the insertion of $n+1$ Jack polynomials given in [MZ11, Appendix C]. The $n=1$ AFLT integral corresponds to the $n=1$ case of this conjecture. The authors note that the formula for general γ fails several consistency checks, and so they believe it requires some minor modifications. Setting $\gamma=1$ also causes the Selberg integral to diverge, and so the conjecture in this case is not necessarily well-defined.

The existence of an A_n Selberg integral with the insertion of $n+1$ Jack polynomials is indicated by the reduction property of Warnaar's A_n integral of Theorem 5.3. As previously observed reduction of the A_n AFLT integral necessarily sets $\mu=0$ in Theorem 5.4 and hence the lower rank integral is not recovered in full.

We have made some minor progress in the case of A_2, with a conjecture regarding the Selberg average with the insertion of three Schur functions. Here, as the A_2 Selberg integral diverges for $\gamma=1$ we consider the average

$$\begin{aligned}&\langle s_\mu[X]s_\eta[Y-X]s_\nu[Y+\beta-1]\rangle^{k_1,k_2}_{\alpha_1,\alpha_2,\beta;1} \\ &\quad:=\lim_{\gamma\to1}\langle P^{(1/\gamma)}_\mu[X]P^{(1/\gamma)}_\eta[Y-X]P^{(1/\gamma)}_\nu[Y+\beta/\gamma-1]\rangle^{k,\ell}_{\alpha_1,\alpha_2,\beta;\gamma}\end{aligned} \tag{5.4.1}$$

where X and Y represent the alphabets of integration variables. As the poles for $\gamma=1$ are independent of the partitions μ,η and ν they will disappear in the ratio and the limit is well-defined. We have gathered some evidence for this conjecture in the case that $s_\eta[Y-X]$ is sufficiently specialised. One also sees that setting $k_1=0$ in (5.4.1) gives the Selberg average

$$\langle s_\eta[Y]s_\nu[Y+\beta-1]\rangle^{0,k_2}_{\alpha_1,\alpha_2,\beta;1}.$$

This reduces to the Selberg average for the AFLT integral. Of course this evidence is not conclusive. We expect that a proof of such an integral formula (5.4.1) lies at the q or q,t-level, where issues of convergence are not as problematic. Such a formula would require a Cauchy-type identity with more free alphabets than the two given by Theorem 3.15. However we have not established such a formula.

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